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Edward J. Stanley Jr.

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OPTIMAL CONTROL SYSTEM DESIGN WITH PRESCRIBED EIGENVALUES VIA CAUER SECOND FORM

by

Edward J. Stanley, Jr.

September 1980

Thesis Advisor: M. J. Goldman

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A method is developed in terms of the Cauer Second Form representation of continued fractions as a means of designing linear single-input-output (SISO) control systems. Optimal closed loop solutions corresponding to a set of prescribed eigenvalues are obtained through minimization of a quadratic performance index. The partitioning method of the Cauer Second Form for system simplification is presented with a simplified inversion technique for the reduced order system.

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Optimal Control System Design With Prescribed Eigenvalues Via Cauer Second Form

by

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Captain, United States Marine Corps
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Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

from the

NAVAL POSTGRADUATE SCHOOL
September 1980

ABSTRACT

A method is developed in terms of the Cauer Second Form representation of continued fractions as a means of designing linear single-input single-output (SISO) control systems. Optimal closed loop solutions corresponding to a set of prescribed eigenvalues are obtained through minimization of a quadratic performance index. The partitioning method of the Cauer Second Form for system simplification is presented with a simplified inversion technique for the reduced order system.

TABLE OF CONTENTS

I. INTRODUCTION- 8

II. PROPERTIES OF CAUER FIRST AND SECOND FORMS- - - - 10

A. CLOSED LOOP SYSTEM IN CAUER FIRST AND SECOND FORMS- - - - - - - - - - - - - - - - - - - 10

1. Cauer First Form- - - - - - - - - - - - - - - 10

2. Cauer Second Form - - - - - - - - - - - - - - 12

B. PHYSICAL INTERPRETATION OF DOMINANT TERMS RESULTING FROM CONTINUED FRACTION EXPANSION - 15

1. Cauer First Form- - - - - - - - - - - - - - - 15

2. Cauer Second Form - - - - - - - - - - - - - - 15

C. CONTINUED FRACTION INVERSION- - - - - - - - - - 16

1. Inversion of Cauer First Form - - - - - - - 21

2. Inversion of Cauer Second Form- - - - - - 25

III. LINEAR SYSTEM ORDER REDUCTION VIA PARTITION OF CAUER SECOND FORM- - - - - - - - - - - - - - - - - 35

A. SIMPLIFYING A TRANSFER FUNCTION - - - - - - - - 37

B. STATE EQUATION SIMPLIFICATION - - - - - - - - - 40

IV. DESIGN OF OPTIMAL LINEAR CONTROL SYSTEMS WITH PRESCRIBED EIGENVALUES - - - - - - - - - - - - - 53

A. INTRODUCTION- 53

B. TRANSFORMATION TO PHASE VARIABLE FORM - - - - 56

C. SIMILAR EIGENVALUES - - - - - - - - - - - - - - - 59

D. DEVELOPMENT OF THE PRESCRIBED EIGENVALUE PROBLEM- - - - - - - - - - - - - - - - 63

1. Evaluation of the State (H) and Linear Transformation $(\underset{\sim}{P})^{\sim}$Matrices- - - - 64

E. DETERMINATION OF THE OPTIMAL CONTROL LAW- - - 71

F. ILLUSTRATIVE EXAMPLES - - - - - - - - - - - - - - 83
  1. Odd Order System (Third) - - - - - - - - - 83
  2. Even Order System (Fourth) - - - - - - - - - 94
  3. Higher Order Example (Seventh)- - - - - - - 105
V. CONCLUSIONS AND DISCUSSION- - - - - - - - - - - - - 120
APPENDIX A - TRANSFORMATION TO STATE VARIABLE FORM- - - 122
APPENDIX B - INVERSION OF CAUER I AND CAUER II FORMS COMPUTER PROGRAM - - - - - - - - - - - 127
APPENDIX C - DETERMINATION OF WEIGHTING MATRIX (Q) FOR PRESCRIBED EIGENVALUES COMPUTER PROGRAM- 131
LIST OF REFERENCES- 149
BIBLIOGRAPHY- 152
INITIAL DISTRIBUTION LIST - - - - - - - - - - - - - - - - 155

LIST OF FIGURES

2.1 Block Diagram of an Nth Order Linear System for Direct Programming - - - - - - - - - - - - - - - - - 11

2.2 Block Diagram Representation of an Nth Order System (Cauer First Form)- - - - - - - - - - - - - - - 13

2.3 Block Diagram Representation of an Nth Order System (Cauer Second Form) - - - - - - - - - - - - - - 14

3.1 Block Diagram of 7th Order System From Continued Fraction Expansion - - - - - - - - - - - - - - 41

3.2 Block Diagram of 2nd Order System From Truncated Continued Fraction Expansion - - - - - - - 42

3.3 Unit Impulse Response Comparison of 7th and 2nd Order Systems- - - - - - - - - - - - - - - 43

3.4 Unit Step Response Comparison of 7th and 2nd Order Systems- - - - - - - - - - - - - - - - - 44

3.5 Block Diagram of Nth Order System Indicating State Variables - - - - - - - - - - - - - - 45

3.6 Nth Order System Simplification to Mth Order - - - 47

3.7 7th Order System Reduction to 2nd Order- - - - - - 49

3.8 Unit Step Response Comparison of 7th and 2nd Order Systems- - - - - - - - - - - - - - - - - 50

3.9 Unit Impulse Response Comparison of 7th and 2nd Order Systems- - - - - - - - - - - - - - - 51

4.1 State Matrix, $\underset{\sim}{H}$, in Cauer Second Form- - - - - - - - 68

4.2 Unit Impulse Response of the Uncompensated (1) and Prescribed Eigenvalue (2) Third Order Systems - 108

4.3 Unit Step Response of the Uncompensated (1) and Prescribed Eigenvalues (2) Third Order Systems- 109

4.4 Unit Impulse Response of the Compensated (1) and Prescribed Eigenvalues (2) Third Order Systems- 110

4.5 Unit Step Response of the Compensated (1) and Prescribed Eigenvalue (2) Third Order System - - - 111

4.6 Unit Impulse Response of the Uncompensated (1) and Prescribed Eigenvalue (2) Fourth Order Systems- 112

4.7 Unit Step Response of the Uncompensated (1) and Prescribed Eigenvalue (2) Fourth Order Systems- 113

4.8 Unit Impulse Response of the Compensated (1) and Prescribed Eigenvalue (2) Fourth Order Systems- 114

4.9 Unit Step Response of the Compensated (1) and Prescribed Eigenvalue (2) Fourth Order Systems - - 115

4.10 Unit Impulse Response of the Uncompensated (1) and Prescribed Eigenvalue (2) Seventh Order Systems- 116

4.11 Unit Step Response of the Uncompensated (1) and Prescribed Eigenvalue (2) Seventh Order Systems- - 117

4.12 Unit Impulse Response of the Compensated (1) and Prescribed Eigenvalue (2) Seventh Order Systems- - 118

4.13 Unit Step Response of the Compensated (1) and Prescribed Eigenvalue (2) Seventh Order Systems- - 119

I. INTRODUCTION

The purpose of this research was to develop an algorithm for obtaining optimal closed loop solutions corresponding to a set of prescribed eigenvalues for single-input single-output (SISO) control systems. It was desired that the algorithm be adaptable to digital computer techniques and unrestricted by system order.

The Cauer Second Form for system dynamics representation was chosen over other alternatives because of the regular pattern of the state and output matrices, and the method of linear system simplification.

In Chapter II, several basic properties of both Cauer First and Second Forms are presented from the theory of continued fractions. A simple and efficient algorithm is also developed for inversion of the continued fraction in either form, independent of Routh's algorithm.

In Chapter III, the method of linear system order reduction based on the Cauer Second Form is amplified. The emphasis on this area was primarily to elucidate the various methods previously employed for system simplification.

The original theoretical work of this thesis is presented in Chapter IV. The objective was to obtain closed

loop solutions corresponding to a prescribed set of eigenvalues. While minimizing a certain cost function, which met desired system characteristics. It is shown, by examples, that the derived algorithm is equally capable of handling systems with multiple and/or complex, as well as, unique sets of real eigenvalues.

The final chapter, Chapter V, presents a discussion of results and suggests areas for future study.

II. PROPERTIES OF CAUER FIRST AND SECOND FORMS

A. CLOSED LOOP SYSTEM IN CAUER FIRST AND CAUER SECOND FORMS

Consider the closed loop transfer function given by:

$$\frac{Y(s)}{U(s)} = \frac{\sum_{L=0}^{n-1} b_i s^i}{s^n + \sum_{i=0}^{n-1} a_i s^i}, \tag{2-1}$$

with block diagram as given in Figure 2.1. Equation (2-1) can be expanded into the Cauer Forms of continued fractions as follows.

1. Cauer First Form

a. Arrange the numerator and denominator polynomials in descending order.

b. Perform continued division.

$$\frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_1 s + b_o}{s^n + a_{n-1}\, s^{n-1} + a_{n-2}\, s^{n-2} + \ldots + a_1 s + a_o} \tag{2-2}$$

$$= \cfrac{1}{h_1 s + \cfrac{1}{h_2 + \cfrac{1}{h_3 s + \cfrac{1}{h_4 + \cdots}}}} \tag{2-3}$$

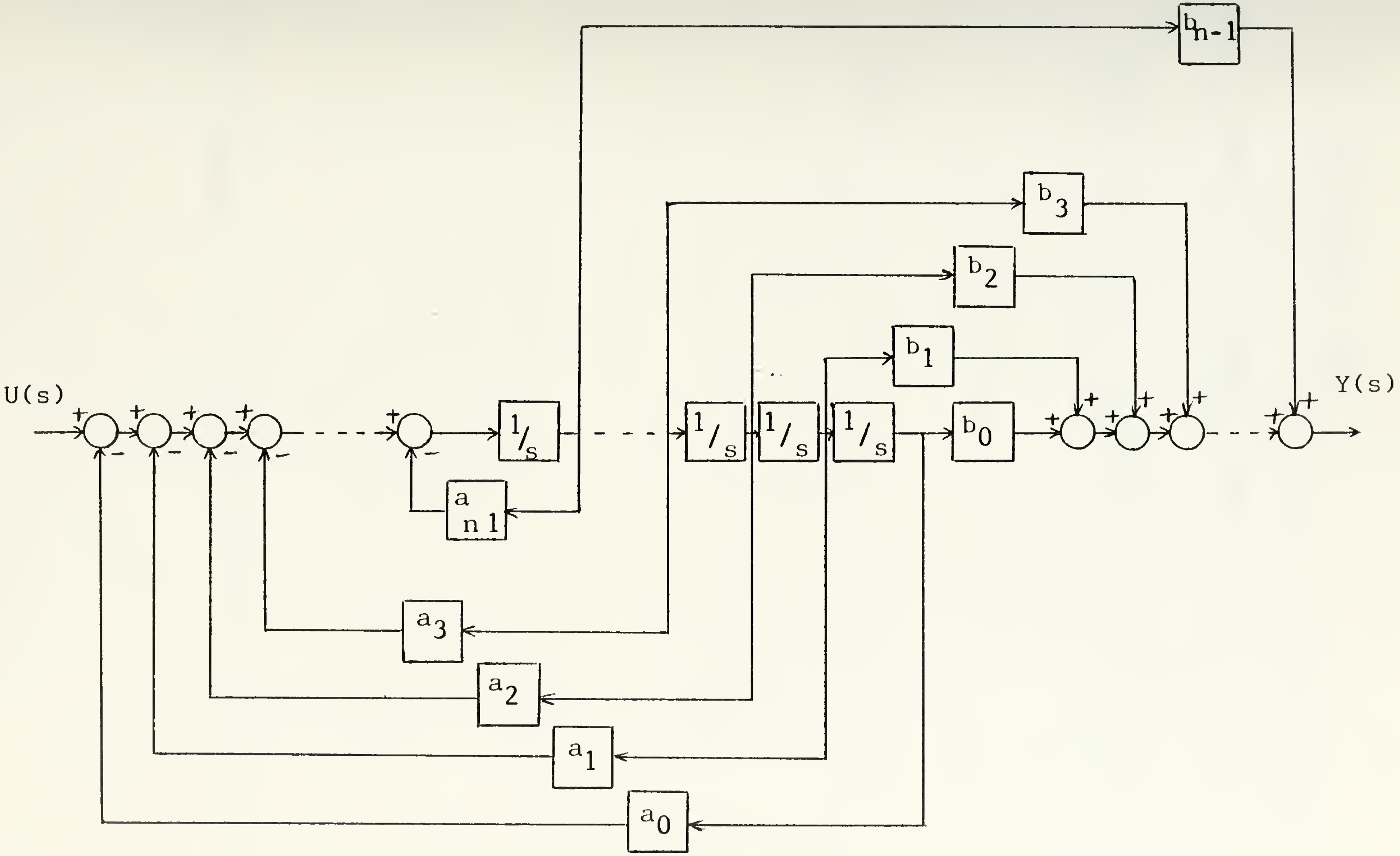

Figure 2.1. Block Diagram of an nth Order Linear System for Direct Programming (Bush Form)

2. Cauer Second Form

a. Invert the numerator and denominator and arrange the polynomials in ascending order.

$$\frac{Y(s)}{U(s)} = \frac{a_o + a_1 s + \ldots . a_{n-2} s^{n-2} + a_{n-1} s^{n-1} + s^n}{b_o + b_1 s + \ldots\ldots\ldots\ldots + b_{n-2} s^{n-2} + b_{N-1} s^{n-1}} \qquad (2\text{-}4)$$

b. Perform continued division.

$$\frac{Y(s)}{U(s)} = \cfrac{1}{h_1 + \cfrac{1}{\frac{h_2}{s} + \cfrac{1}{h_3 + \cfrac{1}{h_4 + \ldots}}}} \qquad (2\text{-}5)$$

or

$$\cfrac{1}{h_1 + \cfrac{S}{h_2 + \cfrac{S}{h_3 + \cfrac{S}{h_4 + \ldots}}}} \qquad (2\text{-}6)$$

Block diagrams of both systems are shown in Figures 2.2 and 2.3.

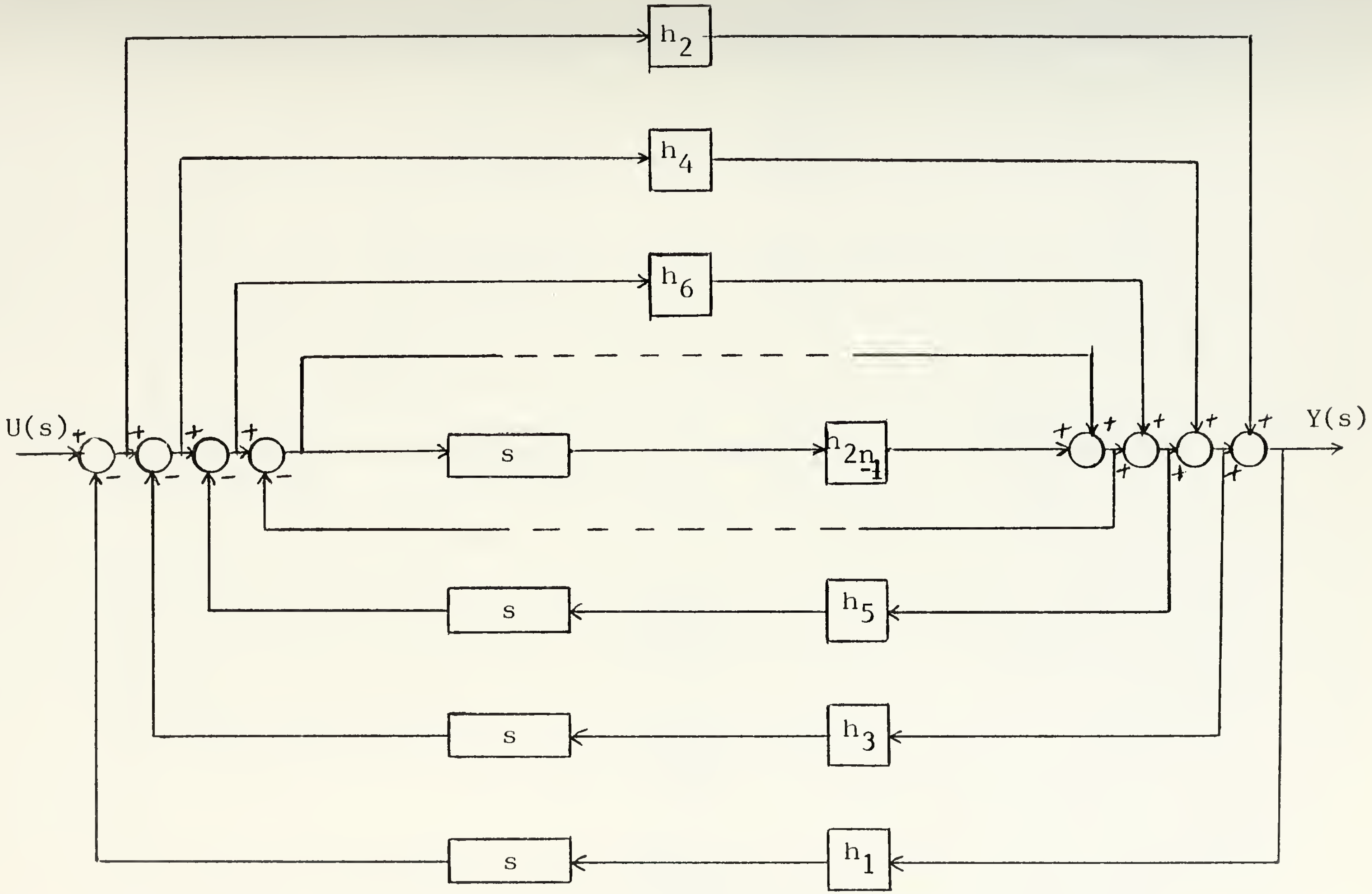

Figure 2.2. Block Diagram Representation of an Nth Order System (Cauer First Form)

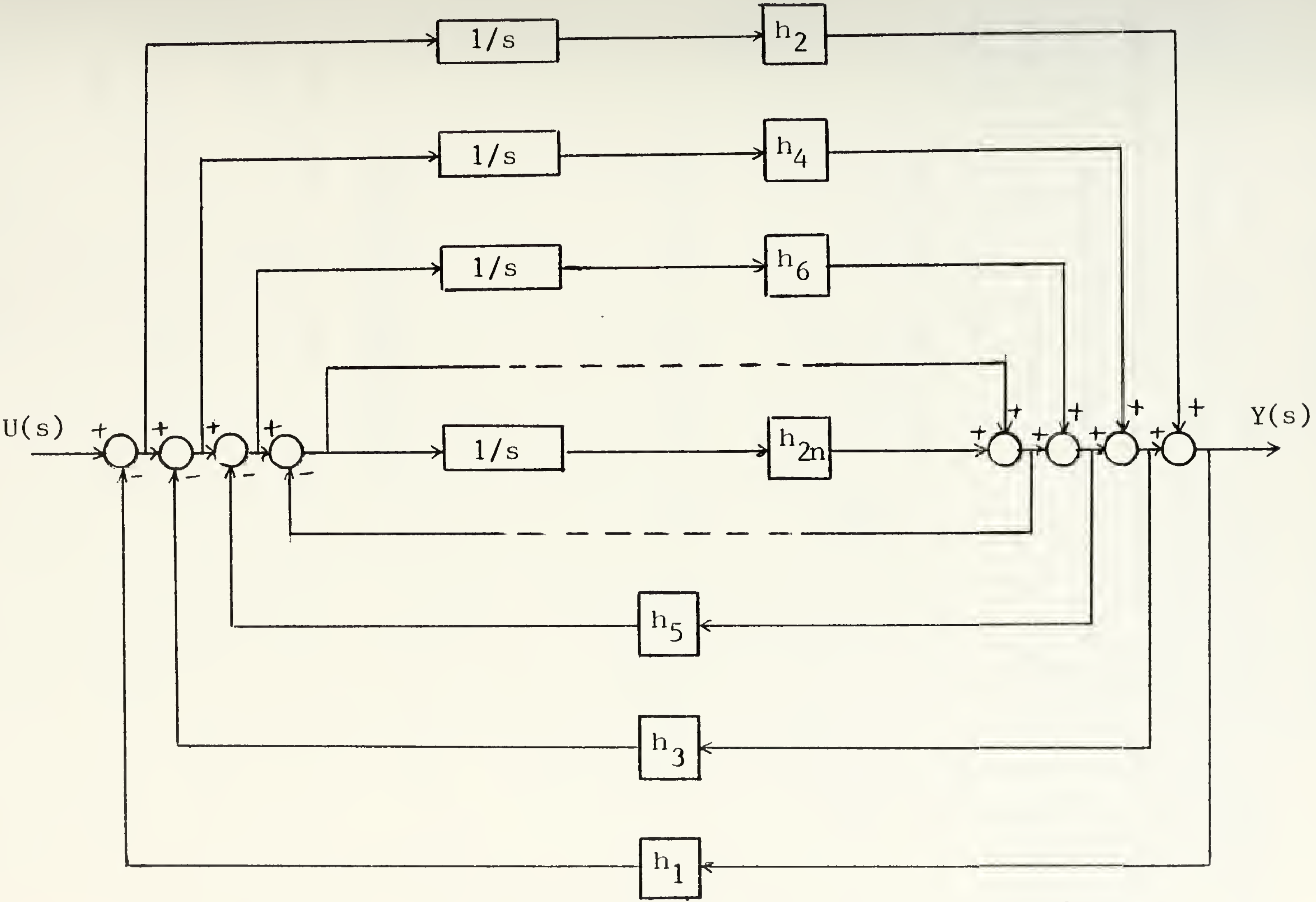

Figure 2.3. Block Diagram Representation of an Nth Order System (Cauer Second Form)

B. PHYSICAL INTERPRETATION OF DOMINANT TERMS RESULTING FROM CONTINUED FRACTION EXPANSION

It is known that the most dominant terms in equations (2-3) and (2-5) are the first quotients, $h_1 s$ and h_1, respectively. A meaningful interpretation for these terms can be accomplished by applying the initial value and final value theorems.[+] Letting $Y(s)/U(s) = F(s)$, by an asymptotic expansion approximation:

1. For Cauer First Form

$$\lim_{t \to 0} f(t) \approx \lim_{s \to \infty} sF(s) \approx \frac{1}{h_1} \tag{2-7}$$

and

$$\lim_{t \to \infty} f(t) \approx \lim_{s \to 0} sF(s) \approx h_2 + h_4 . \tag{2-8}$$

2. For Cauer Second Form

$$\lim_{t \to 0} f(t) \approx \lim_{s \to \infty} sF(s) \approx h_2 + h_4 \tag{2-9}$$

$$\lim_{t \to \infty} f(t) \approx \lim_{s \to 0} sF(s) \approx h_1 . \tag{2-10}$$

[+] $\lim_{t \to \infty} f(t)$ must exist.

Equations (2-7) and (2-10) are of considerable interest since they involve the dominant term, h_1. The implication is that the Cauer First Form emphasizes the initial or transient response of the system; whereas, the Cauer Second Form emphasizes the final or steady state response of the system. In general, the quotients lower in position in the continued fraction expansion have less influence on the performance of the system as a whole, (h_j has less influence than h_i, where $i<j$). Because many systems must meet a set of steady state conditions, the Cauer Second Form will be used for the prescribed eigenvalue problem.

C. CONTINUED FRACTION INVERSION

The theory of continued fractions was first associated with Routh's Algorithm by Wall in 1945, [1] and [2]. The following year Frank [3] extended and modified Wall's work to include complex coefficients. Both, however, applied Routh's algorithm only to continued fraction expansions, not to the problem of inversion.

In 1969, Chen and Shieh [4] developed an algorithm method for converting a continued fraction into a rational fraction of two polynomials. Their method, which makes use of Routh's algorithm, is presented below.

If the elements, h_i, are known for any continued fraction, then the state and output equations can be written immediately from Figures 2.2 or 2.3.

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \vdots \\ \dot{z}_n \end{bmatrix} = - \begin{bmatrix} h_2h_1 & h_4h_1 & h_6h_1 & \cdots h_{2n}h_1 \\ h_2h_1 & h_4(h_1+h_3) & h_6(h_1+h_3) & \cdots h_{2n}(h_1+h_3) \\ h_2h_1 & h_4(h_1+h_3) & h_6(h_1+h_3+h_5)\cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ h_2h_1 & h_4(h_1+h_3) & h_6(h_1+h_3+h_5)\cdots & h_{2n}(h_1+\cdots+h_{2n-1}) \end{bmatrix}$$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \\ z_n \end{bmatrix} + \left| \begin{matrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{matrix} \right| r \qquad (2\text{-}11)$$

$$\underset{\sim}{\dot{z}} = \underset{\sim\sim}{H} \underset{\sim}{z} + \underset{\sim}{D} r \qquad (2\text{-}12)$$

$$C = [h_2 \; h_4 \; h_6 \; \cdots \; h_{2x}] \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \qquad (2\text{-}13)$$

$$C = \underset{\smile}{L} \; \underset{\sim}{z} \qquad (2\text{-}14)$$

The coefficients of the characteristic polynomial, $|s\underset{\sim}{I}-\underset{\sim}{H}|$, become the first row elements of the required Routh array. The next sequence of steps in determining the (j+2)th row of the Routh array is to successively let

$$h_j = 0$$

and

$$h_{j+1} = 0 \tag{2-15}$$

for $j \varepsilon [1,3,5,\ldots 2n\_1]$ and evaluating the remaining $(n-k)x(n-k)$ "H_j" matrix, where $k = (j+1)/2$, up to $k=n-1$, i.e., for n arbitrary and $j=1$; the 3rd row of the Routh array becomes (after h_1 and h_2 are set equal to zero) the coefficients of

$$|s\underset{\sim}{I}-\underset{\sim}{H}_1|,$$

where the $(n-1)x(n-1)$ H_1 matrix is:

$$- \begin{bmatrix} h_4h_3 & h_6h_3 & \ldots\ldots\ldots h_{2n}h_3 \\ h_4h_3 & h_6(h_3+h_5) & \ldots\ldots\ldots h_{2n}(h_3+h_5) \\ \vdots & \vdots & \vdots \\ h_4h_3 & h_6(h_3+h_5) & \ldots\ldots\ldots h_{2n}(h_3+\ldots h_{2n-1}) \end{bmatrix} \tag{2-16}$$

This process repeats until the system state matrix is reduced to a single element, H_{2n-1}, yielding the (2n-1)th row in Routh's array. It is observed that each successive

odd numbered row contains one less element than it's predecessor. By inserting leading zeros in the 3rd, 5th, ..., (2n+1)th row, the matrix, P, is formed.

$$\begin{array}{l} \text{3rd} \\ \text{5th} \\ \text{7th} \\ \\ \\ \\ \text{(2n+1)th} \end{array} \begin{bmatrix} P11 & P12 & P13 & \cdots\cdots\cdots & 1 \\ 0 & P22 & P23 & \cdots\cdots\cdots & 1 \\ 0 & 0 & P33 & \cdots\cdots\cdots & 1 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & & 1 \end{bmatrix} \tag{2-17}$$

The matrix, $\underset{\sim}{P}$, is the linear transformation matrix required to obtain a linear system in Cauer Second Form from phase variable (canonical) form. Continuing, the second row of the Routh array is obtained from the output matrix, L, and the above transformation:

$c = \underset{\sim}{L}\underset{\sim}{z}$ (continued fractions)

$z = \underset{\sim}{P}\underset{\sim}{x}$ (linear transformation)

$y = \underset{\sim}{C}\underset{\sim}{x}$ (output equation, phase variable form)

Therefore,

$$\underset{\sim}{C} = \underset{\sim}{L}\ \underset{\sim}{P}\ . \tag{2-18}$$

$\underset{\sim}{C}$ is an (1xn) vector whose elements are the second row of the Routh array.

Consider the Routh array as an (n+1)x(2n+1) matrix with typical element r_{ij}. The quotients, h_i, of the continued

fraction expansion can be expressed as:

$$h_i = \frac{r_{i1}}{r_{i+1,1}} \tag{2-19}$$

From this relationship and knowledge of how the Routh array is generated, the remaining even numbered rows of the array can be found. The transfer function as a ratio of two polynomials is written as:

$$T(s) = \frac{\sum_{j=1}^{n} r_{2,j} \; s^{j-1}}{\sum_{L=1}^{n+1} r_{1,j} \; s^{i-1}} \tag{2-20}$$

Chen and Shieh [4] contend that this method is the easiest in attaining the inversion. The author disagrees and presents a simpler iterative method based on the inversion technique for the Generalized Cauer Form given by Goldman [5]. The method is equally suited to both Cauer First and Cauer Second Forms, requiring no prior knowledge of Routh's algorithm. Assuming all h_i's are known, and non-zero, in equation (2-3) or (2-5), let:

$$a_i = h_{2i-1} \tag{2-21}$$

$$b_i = h_{2i} \tag{2-22}$$

for $i \in [1,2,\ldots.,n]$.

1. Inversion of Cauer First Form

Initialize two (n+1x1) vectors C and D.

$$\underset{\sim}{C} = [c_0\ c_1\ c_2\ \ldots\ldots\ c_n] \tag{2-23}$$

$$\underset{\sim}{D} = [d_0\ d_1\ d_2\ \ldots\ldots\ d_n] \tag{2-24}$$

to all zeros, except:

$$c_n = b_n \tag{2-25}$$

$$d_{n-1} = a_x \times c_n \tag{2-26}$$

$$d_n = 1. \tag{2-27}$$

The following set of equations are first solved for i=1.

$$c_{n-i+j} = b_{n-i} \times d_{n-i+j} + c_{n-i+j} \tag{2-28}$$

$$d_{n-(i+1)+j} = a_{n-i} \times c_{n-i+j} + d_{n-(i+1)+j}, \tag{2-29}$$

where $j \varepsilon [0,1,2,\ldots,i]$ are substituted in ascending order, and (2-28) is solved before (2-29) for each value of j. Now, let i=2 in equations (2-28) and (2-29) and repeat the same procedure. The index "i" is incremented until i=n-1, and (2-28) and (2-29) are solved as before over the appropriate range of the index "j". The final vectors, $\underset{\sim}{C}$ and $\underset{\sim}{D}$, contain elements which are the coefficients of the numerator and denominator polynomials, respectively, of the transfer

function (or driving point impedance function):

$$T(s) = \frac{\sum_{i=1}^{n} C_i s^{n-i}}{\sum_{j=0}^{n} d_j s^{n-j}} \tag{2-30}$$

Example:

$$T(s) = \frac{10s^2+171s+360}{s^3+71s^2+702s+720} \tag{2-31}$$

By continued fraction division:

$$\begin{array}{lll} 10s^2+171s+360 & \overline{\big)\, s^3+71s^2+702s+720} & .1s \\ & \underline{s^3+17.1s^2+36s} & \\ & \qquad 53.8s^2+666s+720 & \end{array}$$

$$\begin{array}{lll} 53.9s^2+666s+720 & \overline{\big)\, 10s^2+171s+360} & \frac{10}{53.9} \simeq .1855 \\ & \underline{10s^2+123.562s+133.58} & \\ & \qquad 47.438s+226.42 & \end{array}$$

$$\begin{array}{lll} 47.438s+226.42 & \overline{\big)\, 53.9s^2+666s+720} & 1.1362s \\ & \underline{53.9s^2+257.258s} & \\ & \qquad 408.742s+720 & \end{array}$$

$$\begin{array}{lll} 408.742s+720 & \overline{\big)\, 47.438s+226.42} & .115 \\ & \underline{47.438s+82.79} & \\ & \qquad\qquad 143.63 & \end{array} \tag{2-32}$$

$$\begin{array}{lrl} 143.63 & \overline{)408.742s+720} & 2.8458s \\ & \underline{408.742s} & \\ & 720 & \end{array}$$

$$\begin{array}{lrl} 720 & \overline{)143.63} & .1995 \quad , \\ & \underline{143.63} & \\ & 0 & \end{array}$$

Therefore, the transfer function in the form of equation (2-3) is:

$$T(s) = \cfrac{1}{.1s + \cfrac{1}{.1855 + \cfrac{1}{1.1362s + \cfrac{1}{.115 + \cfrac{1}{2.8458s + \cfrac{1}{.1995}}}}}} \qquad (2\text{-}32)$$

with

$$\begin{array}{ll} h_1 = .1 & h_4 = .115 \\ h_2 = .1855 & h_5 = 2.8458 \\ h_3 = 1.1362 & h_6 = .1995. \end{array} \qquad (2\text{-}33)$$

Now, using equations (2-21) and (2-22);

$$\begin{array}{ll} a_1 = h_1 = .1 & b_1 = h_2 = .1855 \\ a_2 = h_3 = 1.1362 & b_2 = h_4 = .115 \\ a_3 = h_5 = 2.8458 & b_3 = h_6 = .1995. \end{array} \tag{2-34}$$

From equations (2-25) through (2-27),

$$\begin{array}{l} c_3 = b_3 = .1995 \\ d_2 = a_3 x c_3 = 2.8458(.1995) = .568 \\ d_3 = 1. \end{array} \tag{2-35}$$

Substituting i=1 into equations (2-28) and (2-29) for j=0:

$$c_2 = b_2 x d_2 + c_2 = .115(.568) + 0 = .0653$$

$$d_1 = a_2 x c_2 + d_1 = 1.1362(.0653) + 0 = .0742$$

for j = 1:

$$c_3 = b_2 c d_3 + c_3 = .115(1) + .1995 = .3145$$

$$d_2 = a_2 x c_3 + d_2 = 1.1362(.3145) + .568 = .9353$$

At this point, j=i, therefore increment index "i":

for i=2, j=0:

$$c_1 = b_1 x d_1 + c_1 = .1855(.0742) + 0 = .0138$$

$$d_0 = a_1 x c_1 + d_0 = .1(.0138) + 0 = .00138$$

for j=1:

$$c_2 = b_1xd_2+c_2 = .1855(.9353) + .0653 = .2388$$

$$d_1 = a_1xc_2+d_1 = .1(.2388) + .0742 = .0981$$

for j=2:

$$c_3 = b_1xd_3+c_3 = .1855(1) + .3145 = .5000$$

$$d_2 = a_1xc_3+d_2 = .1(.5) + .9353 = .9853$$

Now, at the point where j=i, and i=n-1, the transfer function is:

$$T(s) = \frac{.0138s^2+.2388s+.5}{.00138s^3+.0981s^2+.9853s+1} \tag{2-36}$$

Multiplying numerator and denominator by $1/d_o = 720$ yields:

$$T(s) = \frac{10s^2+171s+360}{s^3+71s^2+702s+720} \tag{2-37}$$

2. Inversion of Cauer Second Form

Initialize the two (n+1x1) vectors $\underset{\sim}{C}$ and $\underset{\sim}{D}$, (2-23) and (2-24), to all zeros, except:

$$c_n = b_n \tag{2-38}$$

$$d_{n-1} = 1 \tag{2-39}$$

$$d_n = a_nxc_n \tag{2-40}$$

The following set of equations are first solved for i=1.

$$c_{n-1+j} = b_{n=i} \times d_{n-i+j} + c_{n-i+j+1} \qquad (2\text{-}41)$$

$$d_{n-(i+1)+j} = a_{n-i} \times c_{n-(i+1)+j} + d_{n-i+1} \qquad (2\text{-}42)$$

where $j \in [0,1,2,....,i)^{+}$ are substituted in ascending order, and (2-41) is solved before (2-42) for each value of j.

Next, find d_n according to:

$$d_n = a_{n-i} \times c_n. \qquad (2\text{-}43)$$

Now, let i=2 in equations (2-41) and (2-42) and repeat the same recursive procedure. The index "i" is incremented by one until i = n-1, and for each value of i, (2-41) and (2-42) are iteratively solved over the appropriate range of the index "j". The resulting elements of $\underset{\sim}{C}$ and $\underset{\sim}{D}$ are the coefficients of the numerator and denominator polynomials of the transfer function:

$$T(s) = \frac{\sum_{L=1}^{n} c_i s^{n-i}}{\sum_{j=0}^{n} d_i s^{n-l}} \qquad (2\text{-}30)$$

Using the sample example, (2-31);

$$T(s) = \frac{10s^2+171s+360}{s^3+71s^2+702s+720}$$

[+] for j=i, $c_{n-i+j+1} = 0$.

Place the numerator and denominator terms in ascending order, invert, and perform continued fraction division:

$$\begin{array}{lll}
360+171s+10s^2 & \overline{)\,720+702s+71s^2+s^3} & 2 \\
 & \underline{720+342s+20s^2} & \\
 & \qquad 360s+51s^2+s^3 &
\end{array}$$

$$\begin{array}{lll}
360s+51s+s^3 & \overline{)\,360+171s+10s^2} & 1/s \\
 & \underline{360+51s+s^2} & \\
 & \qquad 120s+9s^2 &
\end{array}$$

$$\begin{array}{lll}
120s+9s^2 & \overline{)\,360s+51s^2+s^3} & 3 \\
 & \underline{360s+27s^2} & \\
 & \qquad 24s^2+s^3 &
\end{array}$$

$$\begin{array}{lll}
24s+s^2 & \overline{)\,120+9s} & 5/s \\
 & \underline{120+5s} & \\
 & \qquad 45 &
\end{array} \tag{2-44}$$

$$\begin{array}{lll}
4 & \overline{)\,24+s} & 6 \\
 & \underline{24} & \\
 & \qquad s &
\end{array}$$

$$\begin{array}{lll}
s & \overline{)\,4} & 4/s \\
 & \underline{4} & \\
 & 0 &
\end{array}$$

The transfer function (2-31), in the form of equation (2-6) is:

$$T(s) = \cfrac{1}{2+\cfrac{s}{1+\cfrac{s}{3+\cfrac{s}{}}}}$$

$$5+\cfrac{s}{6+\cfrac{s}{4}} \tag{2-45}$$

where

$$h_1 = 2 \qquad h_4 = 5$$
$$h_2 = 1 \qquad h_5 = 6$$
$$h_3 = 3 \qquad h_6 = 4 \tag{2-46}$$

For the inversion process, using equations (2-21), (2-22) and (2-46);

$$a_1 = h_1 = 2 \qquad b_1 = h_2 = 1$$
$$a_2 = h_3 = 3 \qquad b_2 = h_4 = 5$$
$$a_3 = h_5 = 6 \qquad b_3 = h_6 = 4 \tag{2-47}$$

and from equations (2-38) through (2-40),

$$c_3 = b_3 = 4$$

$$d_2 = 1$$

$$d_3 = a_3 \times c_3 = 6(4) = 24. \tag{2-48}$$

Now, substituting i = 1 into equation (2-41) and (2-42), for j=0:

$$c_2 = b_2 + c_3 = 5 + 4 = 9$$

$$d_1 = a_2 \times c_1 + d_2 = 3(0) + 1 = 1 \tag{2-49}$$

for j=1;

$$c_3 = b_2 \times d_3 + 0 = 5(24) = 120$$

$$d_2 = a_2 \times c_2 + d_3 = 3(9) + 24 = 51$$

and from equation (2-43):

$$d_3 = a_2 \times c_3 = 3(120) = 360$$

for i=2, j=0:

$$c_1 = b_1 = c_2 = 1 + 9 = 10$$

(2-50)

$$d_0 = a_1 \times c_0 + d_1 = 2(0) + 1 = 1$$

for j=1;

$$c_2 = b_1 \times d_2 + c_3 = 1(51) + 120 = 171$$

$$d_1 = a_1 \times c_1 + d_2 = 2(10 + 51 = 71$$

for j=2;

$$c_3 = b_1 \times d_3 + 0 = 1(360) + 0 = 360$$

$$d_2 = a_1 \times c_2 + d_3 = 2(171) + 360 = 702$$

and from equation (2-43):

$$d_3 = a_1 \times c_3 = 2(360) = 720.$$

Therefore,

$$\underset{\sim}{C} = [0 \quad 10 \quad 171 \quad 360]$$

$$\underset{\sim}{D} = [1 \quad 71 \quad 702 \quad 720], \qquad (2\text{-}51)$$

and the transfer function realized from equation (2-30) is:

$$T(s) = \frac{10s^2+171s+360}{s^3+71s^2+702s+720} \qquad (2\text{-}52)$$

which is the same as (2-31).

This completes the development of the continued fraction inversion algorithms from Cauer First and Second Forms. This iterative procedure is easily seen to be computationally much simpler than Chen and Shieh's method. First, it does not require the need to find the $\underset{\sim}{H}$ matrix, (2-11); and second, it does not necessitate finding the coefficients of n characteristic polynomials of diminishing order. This method is solely based on equation (2-6), enumerating the inversion from bottom to top. As by-product, the entire Routh array appears in the intermediate steps as can be seen from the Cauer Second Form example:

$$\begin{array}{llll} d_3 & d_2 & d_1 & d_0 \\ c_3 & c_2 & c_1 & \\ d_3 & d_2 & d_1 & \\ c_3 & c_2 & & \\ d_3 & d_2 & & \\ c_3 & & & \\ 1 & & & \end{array} = \begin{array}{rrrr} 720 & 702 & 71 & 1 \\ 360 & 171 & 10 & \\ 360 & 51 & 1 & \\ 120 & 9 & & \\ 24 & 1 & & \\ 4 & & & \\ 1 & & & \end{array} \qquad (2\text{-}53)$$

where rows 2(n-i)1 and 2(n-i) are taken from the ith iteration, $i\varepsilon[0,1,2,\ldots,n-1]$. "i=0" implies the rows come from the initialization of $\underset{\sim}{C}$ and $\underset{\sim}{D}$. The last row, the (2n+1)th, is always the single element one.

If a comparison is made between equations (2-20) and (2-30), it is observed that:

$$\hat{c}_i = r_{2,n-i+1}, \quad i\varepsilon[1,2,\ldots,n]$$

$$d_j = r_{1,x-j+1}, \quad j\varepsilon[0,1,2,\ldots,n] \;, \qquad (2\text{-}54)$$

where the c_i's and d_j's are taken from the (n-1)th interation under the index "i".

It is also observed that if the quotients, h_i's, resulting from expansion into Cauer Second (First) Form are used in the inversion algorithm presented for Cauer First (Second) Form, then the $\hat{c}_i$'s and d_i's in the (n-1)th iteration represent the transfer function coefficients in reverse order. This is shown using the preceding example.

From equation (2-46);

$h_1 = 2$ $\quad h_4 = 5$

$h_2 = 1$ $\quad h_5 = 6$

$h_3 = 3$ $\quad h_6 = 4$

and equation (2-47);

$a_1 = h_1 = 2$ $\quad b_1 = h_2 = 1$

$a_2 = h_3 = 3$ $\quad b_2 = h_4 = 5$

$a_3 = h_5 = 6$ $\quad b_3 = h_6 = 4$.

Now, using the inversion scheme for Cauer First Form, from (2-25) through (2-27);

$$c_3 = b_3 = 4$$

$$d_2 = a_3 x c_3 = 6(4) = 24 \tag{2-55}$$

$$d_3 = 1.$$

Making the substitution, i=1, in equations (2-28) and (2-29), for j = 0;

$$c_2 = b_2 \times d_2 + c_2 = 5(24) + 0 = 120$$

$$d_1 = a_2 \times c_2 + d_1 = 3(120) + 0 = 360$$

for j=1, (j=i);

$$c_3 = b_2 \; c \; d_3 + c_3 = 5(1) + 4 = 9$$

$$d_2 = a_2 \times c_3 + d_2 = 3(9) + 24 = 51$$

for i=2, j=0;

$$c_1 = b_1 \times d_1 + c_1 = 1(360) + 0 = 360$$

$$d_0 = a_1 \times c_1 + d_0 = 2(360) + 0 = 720$$

for j=1;

$$c_2 = b_1 \times d_2 + c_2 = 1(51) + 120 = 171$$

$$d_1 = a_1 \times c_2 + d_1 = 2(171 + 360 = 702$$

for j=2, (j=i);

$$c_3 = b_1 \times d_3 + c_3 = 1(1) + 9 = 10$$

$$d_2 = a_1 \times c_3 + d_2 = 2(10) + 51 = 71$$

and d_3 remains unchanged, equal to 1.

Therefore;

$$\underset{\sim}{C} = [0 \quad 360 \quad 171 \quad 10]$$

$$\underset{\sim}{D} = [720 \quad 702 \quad 71 \quad 1], \qquad (2\text{-}56)$$

and the transfer function should be;

$$T(s) = \frac{360s^2+171s+10}{720s^3+702s^2+71s+1}\,. \qquad (2\text{-}57)$$

Since the h_i's from the Cauer Second Form were used in the inversion algorithm from the Cauer Frist Form, the vectors $\underset{\sim}{C}$ and $\underset{\sim}{D}$ require reversing non-zero elements, resulting in:

$$\underset{\sim}{C} = [0 \quad 10 \quad 171 \quad 360]$$

$$\underset{\sim}{D} = [1 \quad 71 \quad 702 \quad 720]\,, \qquad (2\text{-}51)$$

and the correct transfer function is

$$T(s) = \frac{10s^2+171s+360}{s^3+71s^2+702s+720} \qquad (2\text{-}31)$$

A digital computer program (FORTRAN IV) has been written for both Cauer First and Cauer Second Forms and is included as Appendix 3 with documentation.

III. LINEAR SYSTEM ORDER REDUCTION VIA PARTITION OF THE CAUER SECOND FORM

A control system, in general, can consist of many tens or hundreds of elements. In such cases, the problems facing the engineer include: (1) too many variables to efficiently handle; (2) the dimension of the system is too high to comprehend; and (3) the modifications needed to meet required design characteristics are difficult to ascertain. A logical approach is to seek procedural methods which reduce the order of the system to a manageable size yet maintain the basic characteristics of the full dimension model.

A number of different methods for system simplification have been proposed for the reduction of high order dynamic systems to low order models of a more computationally or analytically tractable nature. The approaches used are quite different, but appear to fall into three main groups. The first is to ignore those modes of the original system which contribute little to the overall response. Davison [6] chose to neglect eigenvalues of the original system which are farthest from the origin, retaining only the dominant eigenvalues and hence dominant time constants in the reduced model. The shortcoming of this technique is that many systems do not have any "dominant" roots [7]. Chidambara [8] essentially finds a reduced forcing function

so that the steady state values of the lower order model agree with those of the original system. The consequence of this method is that the approximate model give correct steady state values but incorrect time responses because the reduced forcing function does not excite the modes of the two systems in the same proportions [6]. Marshall [9] proposed the reduction of the state matrix by partitioning it and setting certain rate variables equal to zero in order to maintain the original steady state values. This technique, like Davison's, is based on dominant roots and, therefore, exhibits the same shortcomings.

The second approach is to search in some manner for the coefficients of a set of differential equations of specified order, the response of which is sufficiently close to that of the original system when both are driven by the same inputs. Sinha and Pille [10] proposed a reduction technique based on the iterative application of the matrix pseudo inverse algorithm [11] to determine a model of specified order which minimizes the sum of the squares of the errors between the responses of the original system and the reduced order model to a given input. The main drawback of this method is that the objective function to be minimized is restricted to be the sum of the squares of the errors. Sinha and Bereznai [12] presented a method which minimizes a specified error criterion for a given reduced order model of the original system, based on the pattern-search algorithm

of Hooke and Jeeves [13]. Although this method provides more flexibility than that of Sinha and Pille, it generally requires considerably more computational time due to the poor convergence properties of the pattern-search algorithm.

The third category involves application of the theory of continued fractions. Methods involving this approach have been developed by Chen and Shieh [14].

Sinha and DeBruin [15] and Fellows et al [16] have established the fact that among the methods previously mentioned, the approach by continued fraction expansion is generally the best for linear model simplification.

A. SIMPLIFYING A TRANSFER FUNCTION

The general nature of a control system is that of a low pass filter. Therefore, model simplification should concentrate on the steady state aspects of the response with the transient portion given secondary consideration. As previously shown in Chapter II, the Cauer Second Form exactly characterizes these miens.

Given the nth order original system transfer function:

$$T(s) = \frac{\sum_{i=0}^{n-1} b_i s^i}{s^n + \sum_{j=0}^{n-1} a_j s^j} , \qquad (3-1)$$

where an mth order simplified model of the system (where m is strictly less than n) is desired, the polynomials in

equation (3-1) are rewritten in ascending order;

$$T(s) = \frac{b_0+b_1s+\ldots\quad\ldots b_{x-2}s^{n-2}+b_{n-1}s^{n-1}}{a_0+a_1s+\ldots\quad\ldots a_{x-2}s^{x-2}+a_{n-1}s^{n-1}+s^n} \qquad (3\text{-}2)$$

and expanded into a continued fraction:

$$T(s) = \cfrac{1}{h_1 + \cfrac{s}{h_2 + \cfrac{s}{h_3 + \cfrac{s}{\ddots\; h_{2n-1} + \cfrac{s}{h_{2n}}}}}} \qquad (3\text{-}3)$$

An mth order simplified model is obtained by keeping the first 2m quotients of the expansion, omitting the remainder;

$$T(s) = \cfrac{1}{h_1 + \cfrac{s}{h_2 + \cfrac{s}{\ddots\; h_{2m-1} + \cfrac{s}{h_{2m}}}}} \qquad (3\text{-}4)$$

and performing the inversion of the truncated fraction. The inversion technique presented in Chapter II can be used for this purpose.

Consider the seventh-order system, representing the control system of the pitch rate of a supersonic transport aircraft [10], described by its transfer function:

$$T(s) = \frac{375000(s+.08333)}{s^7+83.635s^6+4097s^5+70342s^4+853703s^3 + 2814271s^2 + 3310875s+281250} \qquad (3\text{-}5)$$

By continued fraction expansion:

$$T(s) = \cfrac{1}{9.00036 + \cfrac{s}{-.486286 + \cfrac{s}{-.036856 + \cfrac{s}{\ddots \; 78496.2032+\cfrac{s}{.00071478}}}}} \qquad (3\text{-}6)$$

Suppose a second-order simplified model is desired. Equation (3-6) has fourteen quotients. For the desired system, the first four quotients are kept with all others discarded. The truncated continued fraction becomes:

$$T_1(s) = \cfrac{1}{9.00036 + \cfrac{s}{-.486286 + \cfrac{s}{-.036856 + \cfrac{s}{.616185}}}} \qquad (3\text{-}7)$$

,

and converted into transfer function form;

$$T_1(s) = \frac{.1299s+.01105}{s^2+1.14644s+.09941} \quad . \qquad (3\text{-}8)$$

The block diagrams of equations (3-5) and (3-8) in the Cauer Second Form are shown in Figures 3.1 and 3.2, respectively. The unit step and impulse responses of the original and simplified systems are compared and shown in Figures 3.3 and 3.4.

B. STATE EQUATION SIMPLIFICATION

The method of system simplification just presented is especially advantageous when converted into state space form. In Figure 2.3, a name for each state variable is given after each integrator, shown in Figure 3.5, from which the state equations and output equation can be directly written.

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} h_2h_1 & h_4h_1 & h_6h_1 & \ldots h_{2n}h_1 \\ h_2h_1 & h_4(h_1+h_3) & h_6(h_1+h_3) & \ldots h_{2n}(h_1+h_3) \\ h_2h_1 & h_4(h_1+h_3) & h_6(h_1+h_3+h_5) & \ldots h_{2n}(h_1+h_3+h_5) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ h_2h_1 & h_4(h_1+h_3) & h_6(h_1+h_3+h_5) & \ldots h_{2n}(h_1+h_3+\ldots h_{2n-1}) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{bmatrix}$$

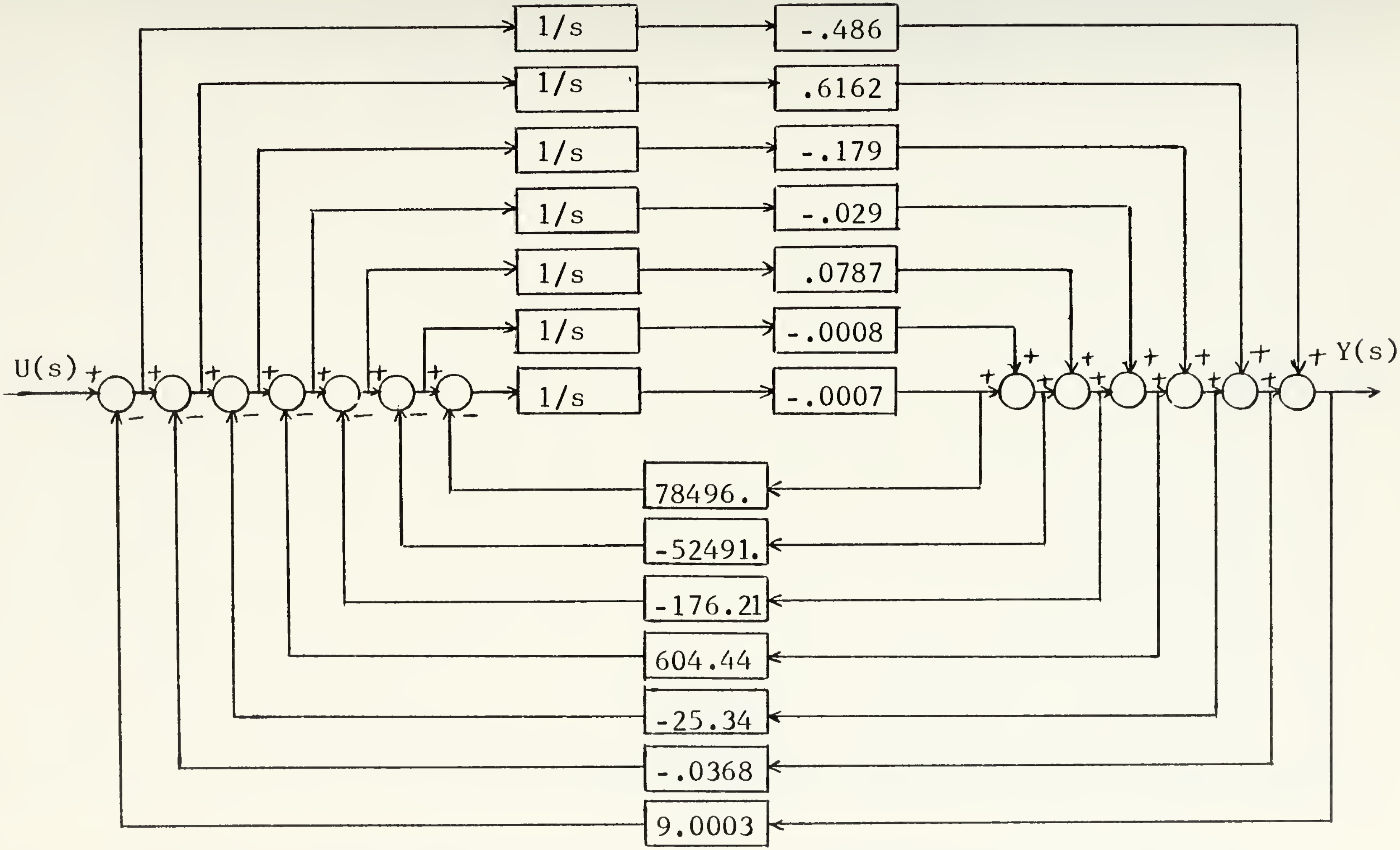

Figure 3.1. Block Diagram of 7th Order System from Continued Fraction Expansion

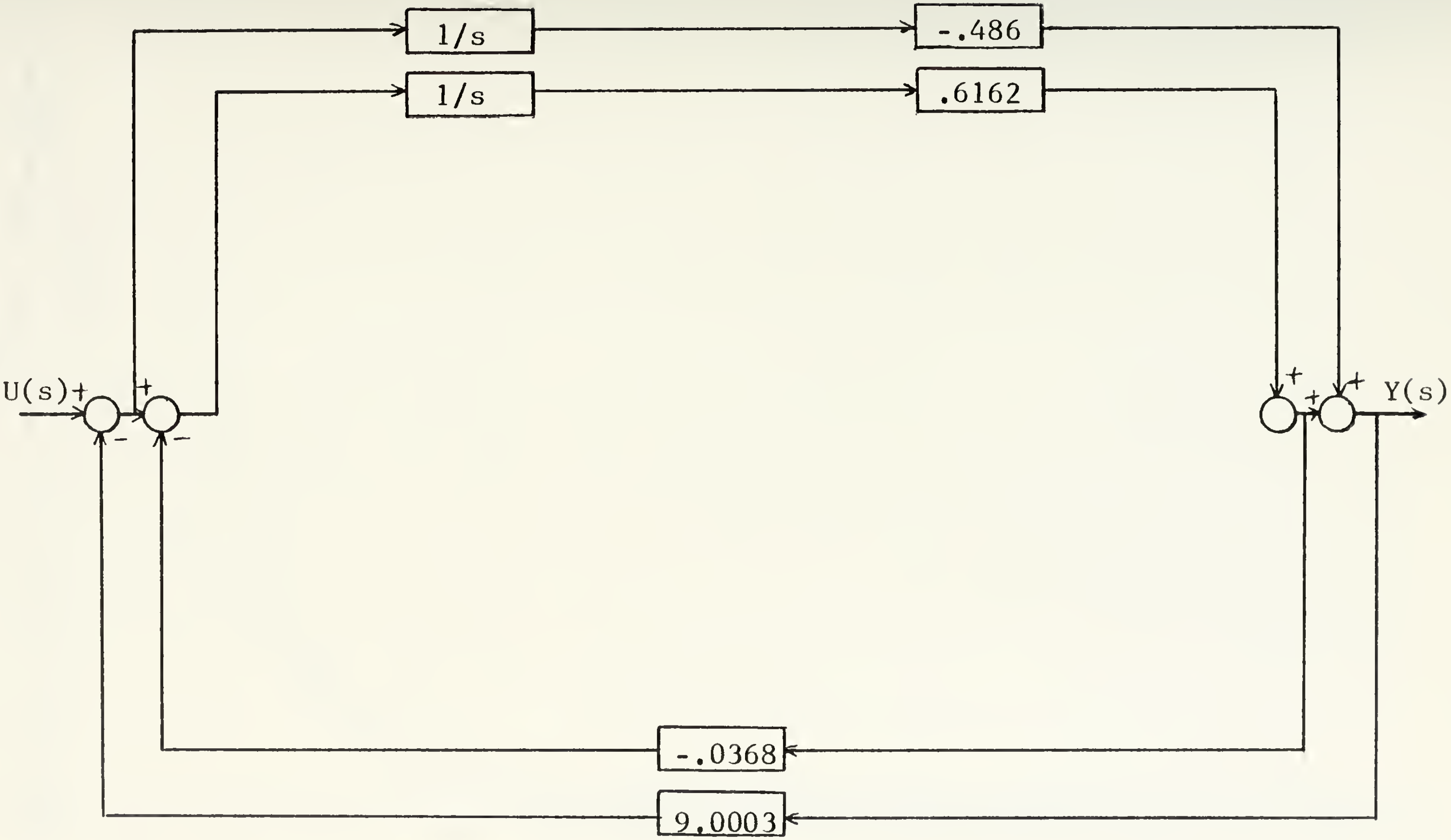

Figure 3.2. Block Diagram of 2nd Order System from Truncated Continued Fraction Expansion

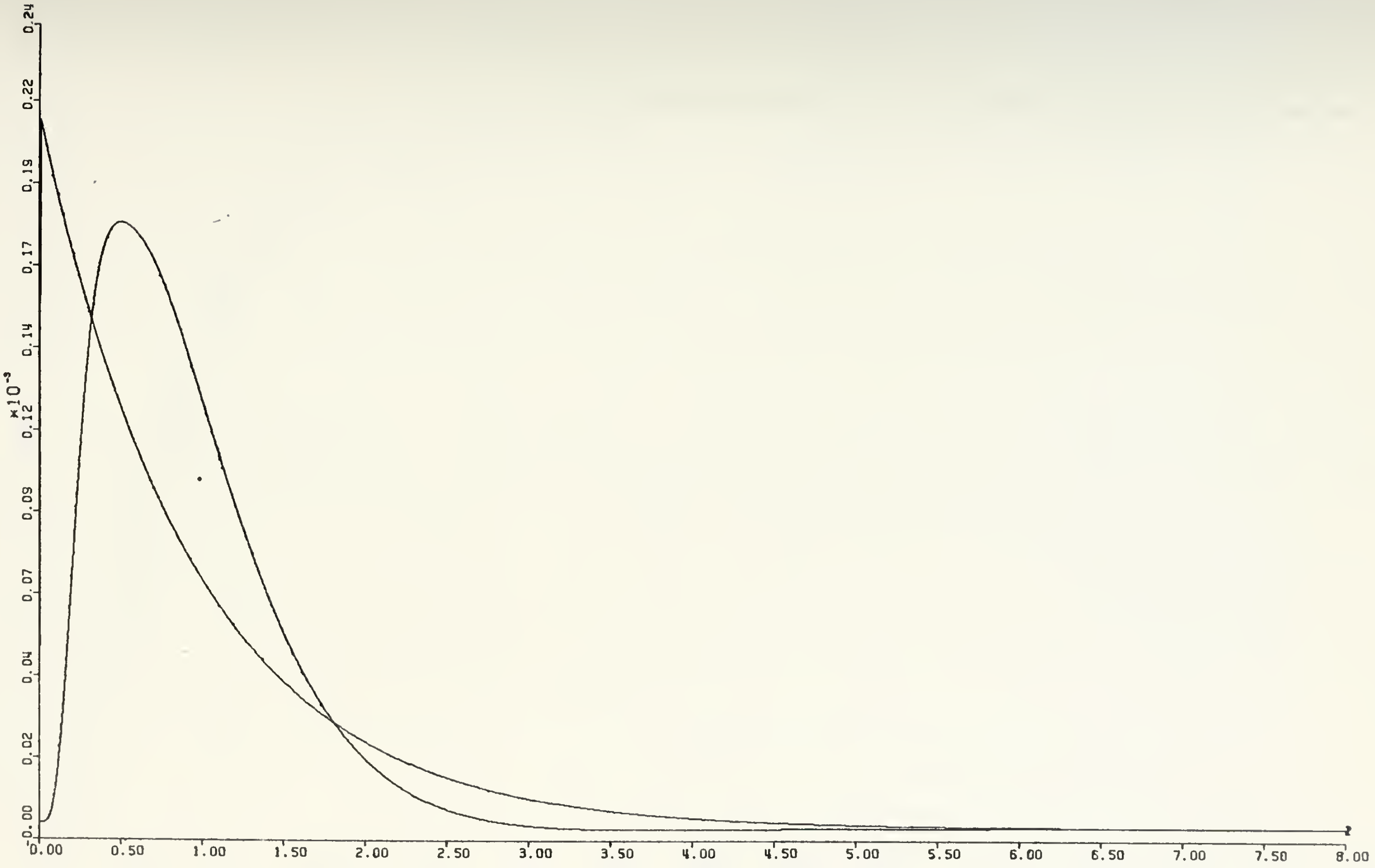

Figure 3.3. Unit Impulse Response Comparison of 7th and 2nd Order Systems

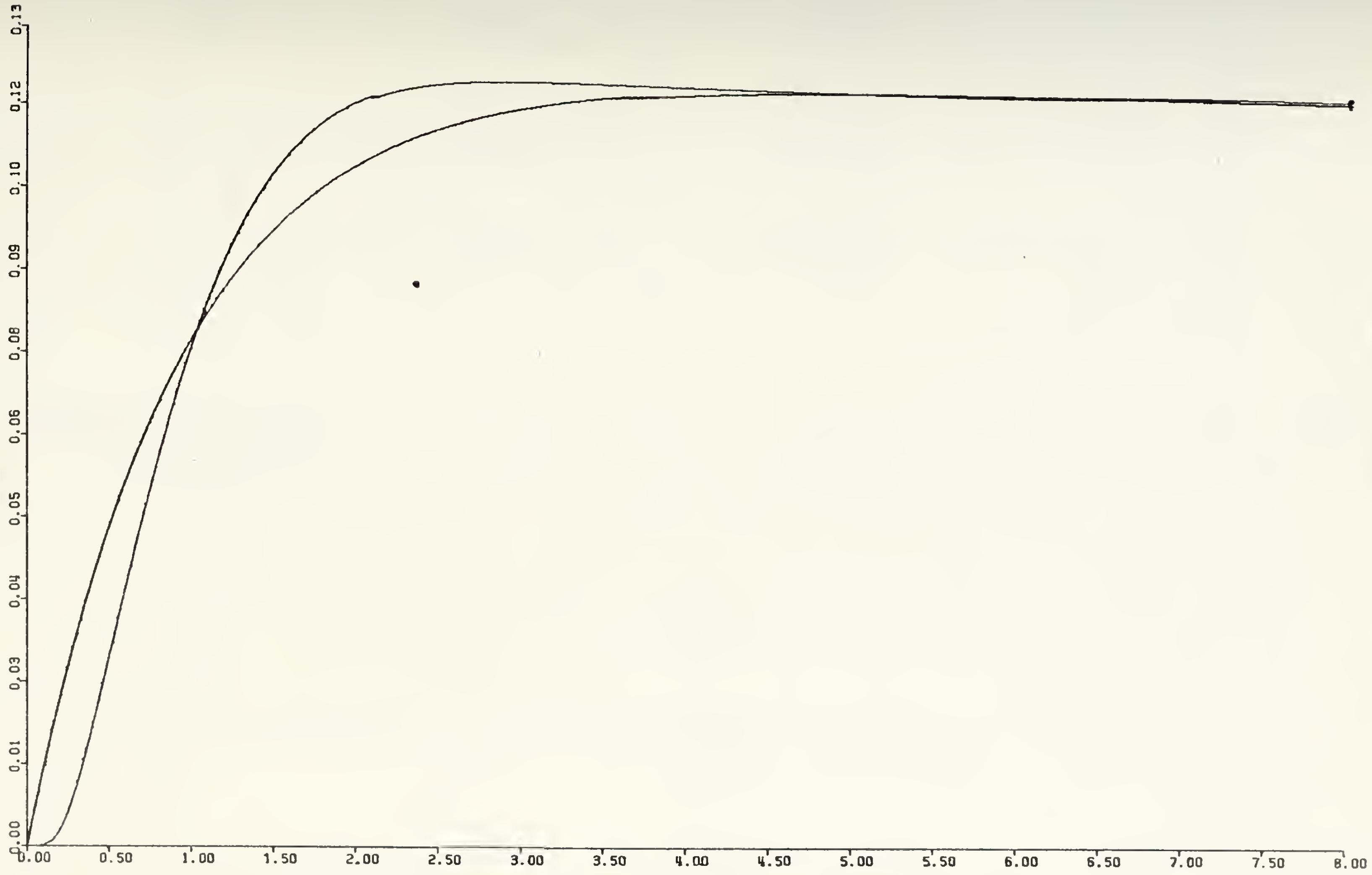

Figure 3.4. Unit Step Response Comparison of 7th and 2nd Order Systems

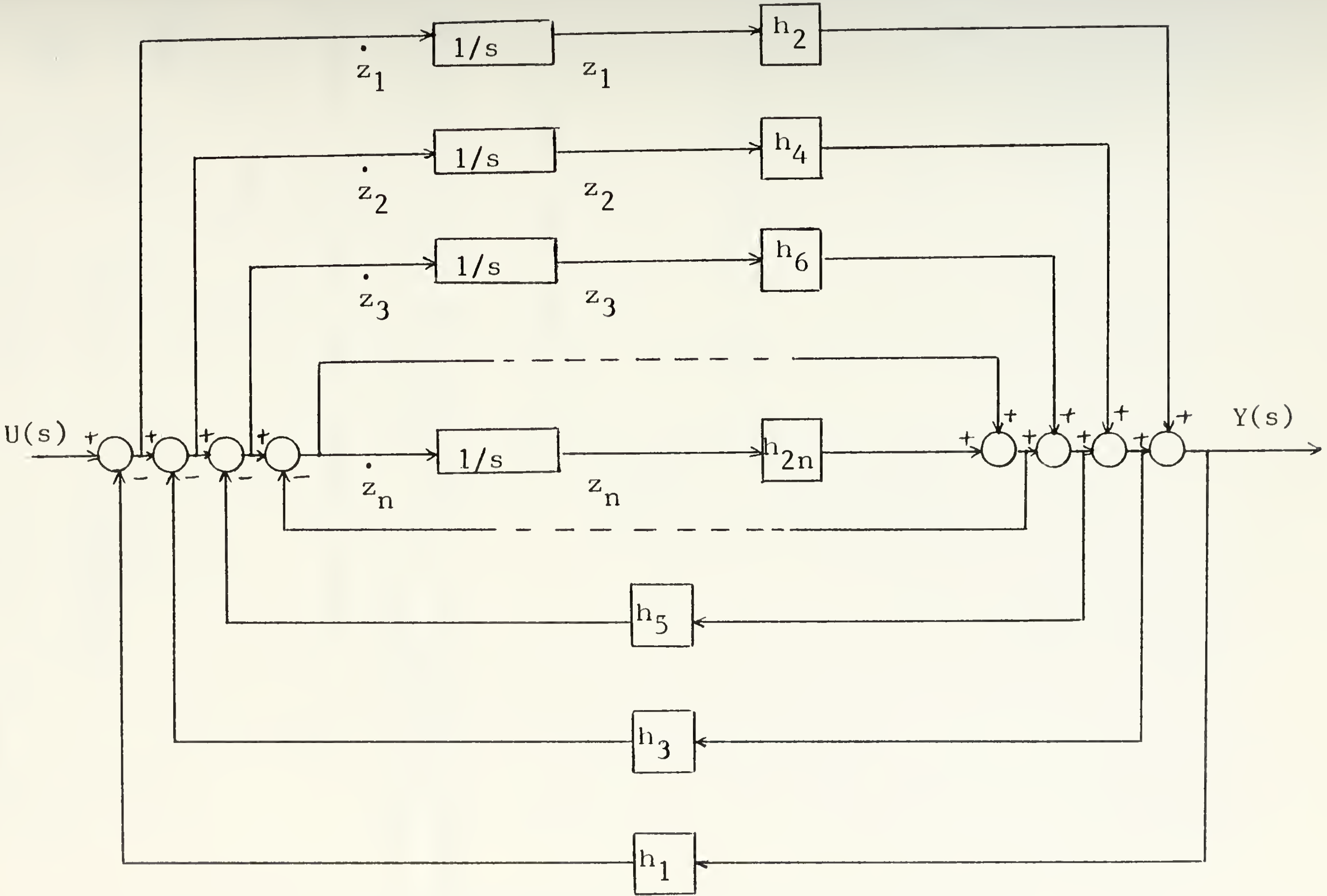

Figure 3.5. Block Diagram of Nth Order System Indicating State Variables

$$+ \begin{bmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ . \\ . \\ 1 \end{bmatrix} r \text{ , and} \qquad (3\text{-}9)$$

$$C = [h_2 \; h_4 \; h_6 \ldots . h_{2n}] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ . \\ . \\ . \\ . \\ z_n \end{bmatrix} ; \qquad (3\text{-}10)$$

$$\dot{\underset{\sim}{z}} = \underset{\sim}{H}\underset{\sim}{z} + \underset{\sim}{D}r \text{ , and } C = \underset{\sim}{L}\underset{\sim}{z} \text{ .} \qquad (3\text{-}11)$$

Simplification of the equations in (3-11) can be achieved by partitioning of $\underset{\sim}{H}$, $\underset{\sim}{D}$ and $\underset{\sim}{L}$, as indicated in Figure 3.6. The resulting mth order system becomes:

$$\dot{\underset{\sim}{z}}_p = \underset{\sim}{Hp} \, \underset{\sim}{z}_p + \underset{\sim}{D}_p r \text{ ,} \qquad (3\text{-}12)$$

where:

$$\underset{\sim}{Hp} = \begin{bmatrix} h_2h_1 & h_4h_1 & \ldots\ldots & h_{2m}h_1 \\ h_2h_1 & h_4(h_1+h_3) & \ldots\ldots & h_{2m}(h_1+h_3) \\ . & . & & . \\ . & . & & . \\ h_2h_1 & h_4(h_1+h_3) & \ldots\ldots & h_{2m}(h_1+\ldots.+j_{2m-1}) \end{bmatrix} , \qquad (3\text{-}13)$$

$$\underset{\sim}{H} = - \left[\begin{array}{cccccc|ccc} h_2h_1 & h_4(h_1+h_3) & \cdots\cdots & h_{2m}h_1 & & \cdots\cdots & & \cdots\cdots & h_{2n}h_1 \\ h_2h_1 & h_4(h_1+h_3) & \cdots\cdots & h_{2m}(h_1+h_3) & & \cdots & & \cdots & h_{2n}(h_1+h_3) \\ \vdots & \vdots & \ddots & \vdots & & & & & \vdots \\ h_2h_1 & h_4(h_1+h_3) & \cdots & h_{2m}(h_1+..+h_{2m-1}) & & \cdots & & \cdots & h_{2n}(h_1+..+h_{2m-1}) \\ \hline \vdots & \vdots & & \vdots & & & & \ddots & \vdots \\ h_2h_1 & h_4(h_1+h_3) & \cdots & h_{2m}(h_1+..+h_{2m-1}) & & \cdots & & \cdots & h_{2n}(h_1+..+h_{2n-1}) \end{array}\right] \qquad \underset{\sim}{D} = \left[\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \\ \hline \vdots \\ 1 \end{array}\right]$$

$$\underset{\sim}{L} = \left[\begin{array}{cccc|cc} h_2 & h_4 & \cdots\cdots & h_{2m} & \cdots & h_{2n} \end{array}\right]$$

Figure 3.6. Nth Order System Simplification to Mth Order

$$\underset{\sim}{D}_p = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} ; \qquad (3\text{-}14)$$

and $c_p = \underset{\sim}{h}_p \, \underset{\sim}{z}_p$, $\qquad (3\text{-}15)$

$$c_p = [h_2 \; h_4 \; \ldots . \; h_{2m}] \begin{bmatrix} z_{p1} \\ z_{p2} \\ \\ z_{pm} \end{bmatrix} . \qquad (3\text{-}16)$$

As an example, consider the seventh order system described by the transfer function:

$$\frac{C(s)}{R(s)} = \frac{1441.53s^3+78319s^2+525286.125s+607693.25}{s^7+112.04s^6+3755.92s^5+39736.73s^4 + 363650.56s^3+759894.19s^2+683656.25s+617497.375} \qquad (3\text{-}17)$$

Arranging the polynomials in ascending order and expanding into the continued fraction yields:

$$\underset{\sim}{H} = \left[\begin{array}{cc|ccccc} -4.1195 & 3.8651 & 0.2516 & 0.3044 & -0.3079 & 0.0063 & -0.0001 \\ -4.1195 & 3.6097 & 0.2350 & 0.2843 & -0.2876 & 0.0059 & -0.0001 \\ \cline{1-2} -4.1195 & 3.6097 & -2.5012 & -3.0261 & 3.0610 & -0.0629 & 0.0007 \\ -4.1195 & 3.6097 & -2.5012 & -14.307 & 14.473 & -0.2974 & 0.0036 \\ -4.1195 & 3.6097 & -2.5012 & -14.307 & 7.1141 & -0.1462 & 0.0017 \\ -4.1195 & 3.6097 & -2.5012 & -14.307 & 7.1141 & -57.414 & 0.6982 \\ -4.1195 & 3.6097 & -2.5012 & -14.307 & 7.1141 & -57.414 & -44.421 \end{array}\right] \qquad \underset{\sim}{D} = \left[\begin{array}{c} 1 \\ 1 \\ \hline 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right]$$

$$\underset{\sim}{L} = \left[\begin{array}{cc|ccccc} 4.0541 & 3.8037 & -0.2476 & -0.2996 & 0.3031 & -0.0063 & 0.0001 \end{array}\right]$$

Figure 3.7. 7th Order System Reduction to 2nd Order

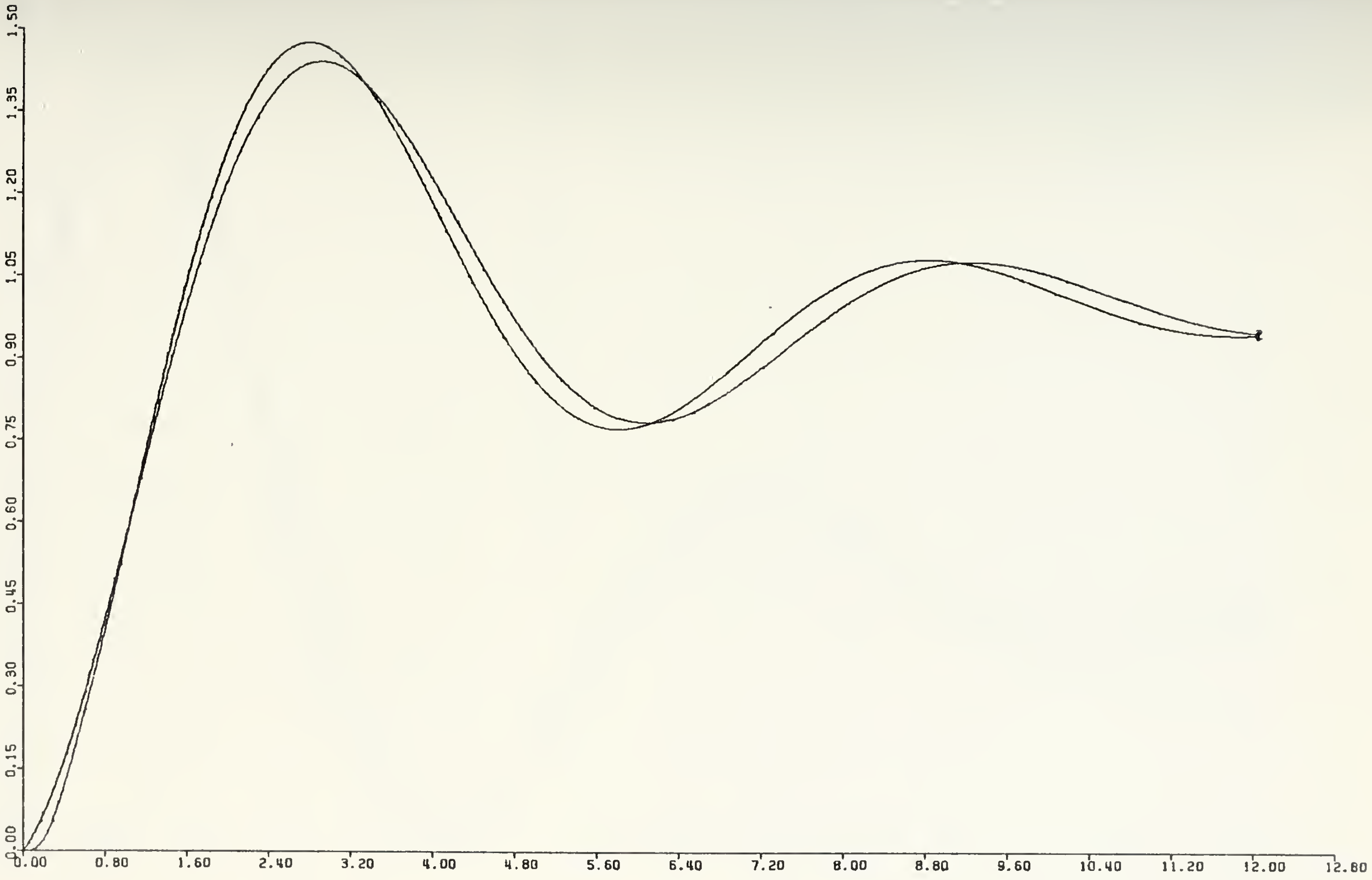

Figure 3.8. Unit Step Response Comparison of 7th and 2nd Order Systems

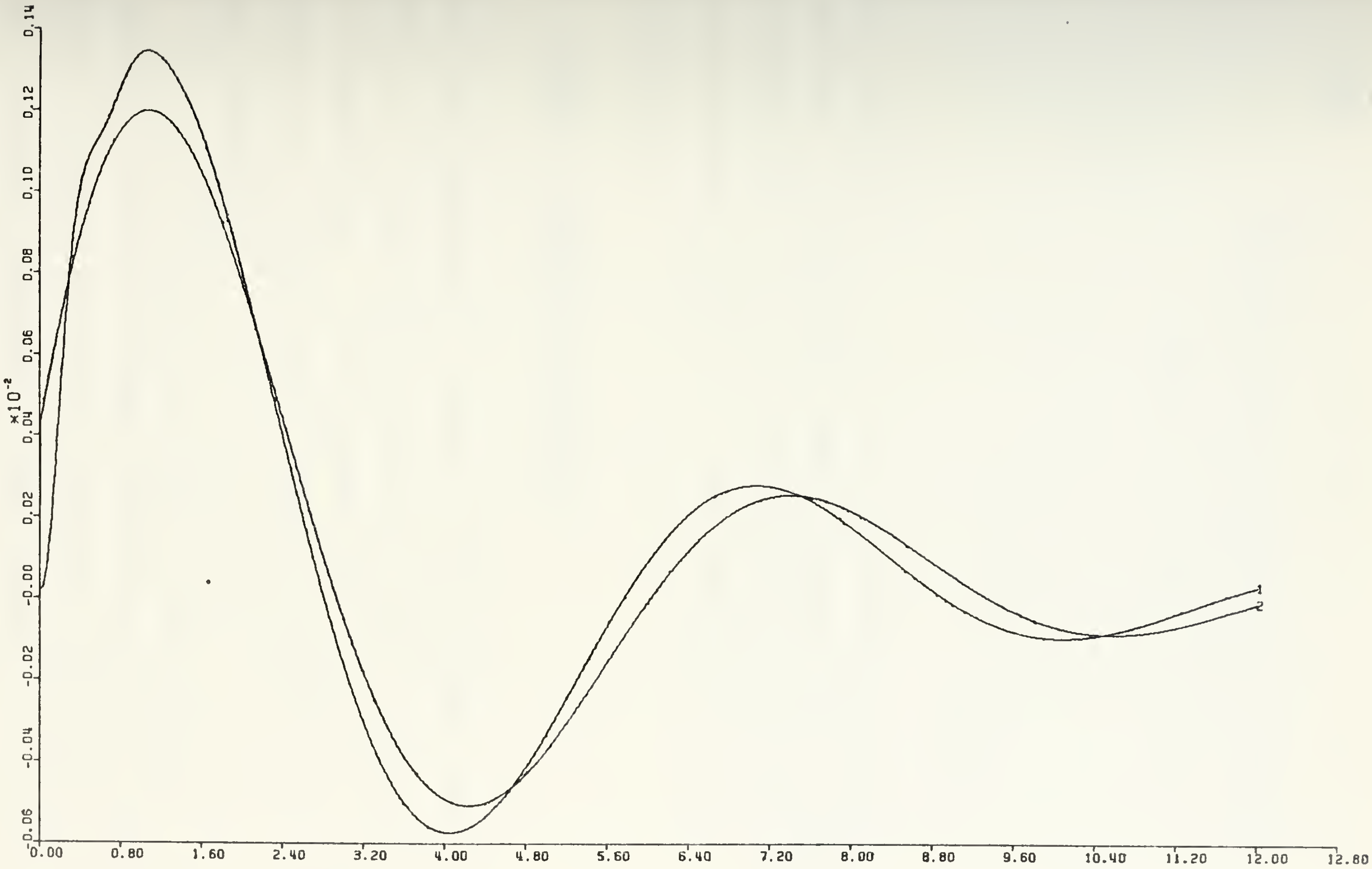

Figure 3.9. Unit Impulse Response Comparison of 7th and 2nd Order Systems

$$\frac{C(s)}{R(s)} = \cfrac{1}{1.016133 + \cfrac{s}{4.054112 + \cfrac{s}{-.067134 + \cfrac{s}{\ddots \, 595660.646 + \cfrac{s}{.0000757}}}}} , \qquad (3\text{-}18)$$

and equations (3-9) and (3-10) are formulated from the quotients. A simplified model of second order is desired, therefore, the state and output equations are partitioned as indicated in Figure 3.7. The simplified transfer function is:

$$\frac{Cp(s)}{RP(s)} = \frac{.250367s+1.035264}{s^2+.509768s+1.051966} \qquad (3\text{-}19)$$

Unit step and impulse responses of the original seventh and simplified second order systems are shown in Figures 3.8 and 3.9 respectively. It is observed that the results of expressing the seventh-order system by a second order model are satisfactory.

It should be pointed out that a stable transfer function may produce an unstable simplified function because the method of continued fraction expansion approximation does not necessarily guarantee a stable system.

IV. DESIGN OF OPTIMAL LINEAR CONTROL SYSTEMS WITH PRESCRIBED EIGENVALUES

A. INTRODUCTION

Consider the control of a plant with dynamics described by a set of first order, time-invariant linear differential equations of the form

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{B}u, \qquad (4\text{-}1)$$

where $\underset{\sim}{x}$ is the nth-order state vector, $\underset{\sim}{A}$ is the (nxn) plant matrix, u is the scaler control and $\underset{\sim}{B}$ is the (nxl) input matrix. The output is defined as

$$y = \underset{\sim}{C}\underset{\sim}{x} , \qquad (4\text{-}2)$$

where $\underset{\sim}{C}$ is the (lxn) output matrix.

A linear feedback control law is assumed, and of the form

$$u = \underset{\sim}{G}^{*}\underset{\sim}{x}.^{+} \qquad (4\text{-}3)$$

There are mainly two separate approaches in the determination of the feedback control matrix, $\underset{\sim}{G}^{*}$, corresponding to the system under consideration; 1 - optimal control and 2 - modal.

In the optimal control approach a performance index is considered which is to be minimized in the design of a

[+] All states are available or an observer or Kalman filter is used to obtain the unknown states.

system. Assuming a performance index can be defined that represents most of the design requirements, "the solution to the optimal control problem can usually be obtained only by numerical methods that yield solutions to only a particular problem" [17]. If solutions are sought to more than one numerical problem, simple performance indices must be defined, which often do not satisfy many of the design requirements. Therefore, the choice of a performance index must fall somewhere between a realistic criterion and one that is mathematically tractable.

A quadratic performance index will be considered as a criterion for designing linear systems, of the form

$$J = \tfrac{1}{2} \int_{0}^{\infty} [\underset{\sim}{x}^{T} \underset{\sim}{Q} \underset{\sim}{x} + Ru^{2}]dt \; , \qquad (4\text{-}4)$$

where $\underset{\sim}{Q}$ is a diagonal non-negative definite (nxn) matrix, and R is a positive scalar.

In the modal approach, $\underset{\sim}{G}^{*}$ is chosen so that the closed loop system achieves the prescribed eigenvalues. Equations (1) and (3) together yield

$$\dot{\underset{\sim}{x}} = (\underset{\sim}{A} + \underset{\sim}{B}\underset{\sim}{G}^{*})\underset{\sim}{x} \; .$$

If $\underset{\sim}{Q}$ and R are given in the optimal control approach, then the eigenvalues of the closed loop system are uniquely determined, which may not realize the required performance characteristics or desired degree of stability for the

system. Using the modal control approach, a feedback control matrix can be found that will give the system the desired eigenvalues. This matrix is usually not unique, and it is not possible in a practical sense to find one that is "better" than it's predecessors, since a performance measure is generally not known that corresponds to a given feedback control matrix. Therefore, it is necessary to find a method for determining the matrix, G*, that simultaneously satisfies the desired eigenvalues and minimizes a given performance index.

In addressing this problem, Chang [18] and Tyler and Tuteric [19] have applied the root locus method to single-input, single-output and multivariable systems, respectively. The method lacks a rational computational procedure for determining the elements of the weighting matrix, $\underset{\sim}{Q}$, to meet a set of prescribed closed loop eigenvalues. Anderson and Moore [20] presented a restrictive method whereby a set of eigenvalues may be located to the left of a line parallel to the imaginary axis in the complex plane. Chen and Shieh [14] presented a method using sensitivity analysis. Solheim's [17] method of a diagonalized (decoupled) system becomes complicated when the system contains either complex or multiple eigenvalues. Systems that cannot be diagonalized add further to the complication of the method.

The method developed here takes advantage of the properties of the Cauer Second Form, is approached in a simplistic manner, and is easy to implement computationally.

B. TRANSFORMATION TO PHASE VARIABLE FORM

Consider an nth order linear system of the form

$$\dot{\underset{\sim}{e}} = \underset{\sim\sim}{Se} + \underset{\sim}{T}f \ , \qquad (4\text{-}5)$$

with output

$$d = \underset{\sim\sim}{We} \ . \qquad (4\text{-}6)$$

Silverman [21], et al., have shown that if the system is controllable, then there exists a non-singular transformation matrix which takes an arbitrary state variable system to phase variable (canonical) form. (see Appendix A)

$$\dot{\underset{\sim}{x}} = \underset{\sim\sim}{Ax} + \underset{\sim}{B}u \qquad (4\text{-}1)$$

$$y = \underset{\sim}{C}\,\underset{\sim}{x} \ , \qquad (4\text{-}2)$$

where

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 & 0 \ldots\ldots\ldots\ldots\ldots & 0 \\ 0 & 0 & 1 \quad 0 \ldots\ldots\ldots & 0 \\ \cdot & \cdot & 0 \quad 1 \ldots\ldots\ldots & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 \ldots\ldots\ldots\ldots\ldots & 1 \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} \ldots\ldots\ldots\ldots & \alpha_{nn} \end{bmatrix}$$

$$\underset{\sim}{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \qquad \underset{\sim}{C} = [\beta_1 \; \beta_2 \; \beta_3 \; \ldots\ldots \; \beta_n] \; .$$

If the system is not controllable, the phase variable (canonical) form may still be obtained from the system transfer function

$$\frac{Y(s)}{U(s)} = \frac{\sum_{i=1}^{n} \beta_i s^{i-1}}{s^n + \sum_{i=1}^{n} \alpha_{ni} s^{i-1}} \; . \tag{4-7}$$

Once the system is in phase variable form, Chen and Shieh [22] have shown that the equivalent system in Cauer Second Form is easily written as

$$\dot{\underset{\sim}{z}} = \underset{\sim}{H}\underset{\sim}{z} + \underset{\sim}{V}u \tag{4-8}$$

$$y = \underset{\sim}{C}^{*}\underset{\sim}{z} \; , \tag{4-9}$$

where the two forms are related by a linear nonsingular transformation matrix, $\underset{\sim}{P}$,

$$\underset{\sim}{z} = \underset{\sim}{P}x \; . \tag{4-10}$$

The matrix $\underset{\sim}{P}$ is an (nxn) upper triangular matrix.

The performance measure under consideration becomes

$$J = \tfrac{1}{2} \int_{0}^{\infty} [\underset{\sim}{z}^T \underset{\sim}{\tilde{Q}} \underset{\sim}{z} + u^T R u] dt \qquad (4\text{-}11)$$

where

$$\underset{\sim}{\tilde{Q}} = (\underset{\sim}{P}^{-1})^T \underset{\sim}{Q} \underset{\sim}{P}^{-1} .$$

From optimal control theory, the Hamiltonian

$$H = \tfrac{1}{2}[\underset{\sim}{z}^T \underset{\sim}{\tilde{Q}} \underset{\sim}{z} + u^T R u] + \underset{\sim}{\hat{p}}^T [\underset{\sim}{H} \underset{\sim}{z} + \underset{\sim}{V} u] , \qquad (4\text{-}12)$$

where $\underset{\sim}{P}$ is the set of Lagrange Multipliers (also called the costate or adjoint vector). For the Hamiltonian to be globally minimized, assuming no bounds on admissible states and control values, it is necessary that $\partial H/\partial u = 0$ and $\partial^2 H/\partial u^2 > 0$.

$$\partial H/\partial u = Ru + \underset{\sim}{V}^T \underset{\sim}{\hat{p}} = 0 \qquad (4\text{-}13)$$

implies

$$u^* = -R^{-1} \underset{\sim}{V}^T \underset{\sim}{\hat{p}} , \text{ and} \qquad (4\text{-}14)$$

$$\partial^2 H/\partial u^2 = R > 0 , \qquad (4\text{-}15)$$

since R was defined as a positive scalar. Included in the necessary conditions for optimality are

$$\dot{\underset{\sim}{z}} = \underset{\sim}{H} \underset{\sim}{z} + \underset{\sim}{V} u^* \qquad (4\text{-}16)$$

$$\partial H/\partial z = -\dot{\hat{\underset{\sim}{p}}} = \underset{\sim}{\tilde{Q}} \underset{\sim}{z} + \underset{\sim}{H}^T \underset{\sim}{\hat{p}} \quad . \qquad (4\text{-}17)$$

Combining equations (4-14), (4-16) and (4-17) yields a set of 2n linear differential equations forming the canonical system in Cauer Second Form.

$$\begin{bmatrix} \dot{\underset{\sim}{z}} \\ \dot{\hat{\underset{\sim}{p}}} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{H} & \underset{\sim}{V}R^{-1}\underset{\sim}{V}^T \\ -\tilde{\underset{\sim}{Q}} & -\underset{\sim}{H}^T \end{bmatrix} \begin{bmatrix} \underset{\sim}{z} \\ \hat{\underset{\sim}{p}} \end{bmatrix} \tag{4-18}$$

It remains to be shown that this form is useful in obtaining optimal closed loop solutions that correspond to a set of prescribed eigenvalues.

C. SIMILAR EIGENVALUES

Consider the 2nth order cononical system in phase variable form:

$$\begin{bmatrix} \dot{\underset{\sim}{x}} \\ \dot{\underset{\sim}{p}} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A} & -\underset{\sim}{B}R^{-1}\underset{\sim}{B}^T \\ -\underset{\sim}{Q} & -\underset{\sim}{A}^T \end{bmatrix} \begin{bmatrix} \underset{\sim}{x} \\ \underset{\sim}{p} \end{bmatrix} . \tag{4-19}$$

It is known that this system possesses n eigenvalues with negative real parts and n eignevalues with positive real parte, and that they are located symmetrically about the imaginary axis in the complex plane [17]. The eigenvalues of the optimal feedback system

$$\dot{\underset{\sim}{x}} = (\underset{\sim}{A} + \underset{\sim}{B}\underset{\sim}{G}^*)\underset{\sim}{x} \tag{4-20}$$

are identical to those with negative real parts in the canonical system. It is possible, therefore, to focus on the canonical system where the dependence eigenvalues on the weighting matrices $\underset{\sim}{Q}$ and R can be studied without solving the matrix Riccati equation. The problem is in determining a weighting matrix, $\underset{\sim}{Q}$, such that the system attains the prescribed set of eigenvalues.

The canonical system in Cauer Second Form can be obtained from the phase variable form using the linear transformation

$$\underset{\sim}{z} = \underset{\sim}{P}\,\underset{\sim}{x} \;, \tag{4-10}$$

where $\underset{\sim}{P}$ is a nonsingular matrix. We have

$$\begin{bmatrix} \dot{\underset{\sim}{x}} \\ \dot{\underset{\sim}{p}} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A} & -\underset{\sim}{B}\underset{\sim}{R}^{-1}\underset{\sim}{B}^{T} \\ -\underset{\sim}{Q} & -\underset{\sim}{A}^{T} \end{bmatrix} \begin{bmatrix} \underset{\sim}{x} \\ \underset{\sim}{p} \end{bmatrix} \tag{4-19}$$

$$\begin{bmatrix} \dot{\underset{\sim}{z}} \\ \dot{\hat{\underset{\sim}{p}}} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{P} & \underset{\sim}{0} \\ \underset{\sim}{0} & (P^{-1})^{T} \end{bmatrix} \begin{bmatrix} \underset{\sim}{A} & -\underset{\sim}{B}\underset{\sim}{R}^{-1}\underset{\sim}{B}^{T} \\ -\underset{\sim}{Q} & -A^{T} \end{bmatrix} \quad \times \tag{4-20}$$

$$\begin{bmatrix} \underset{\sim}{P}^{-1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{P}^{T} \end{bmatrix} \begin{bmatrix} \underset{\sim}{z} \\ \hat{\underset{\sim}{p}} \end{bmatrix} \tag{4-21}$$

$$\begin{bmatrix} \dot{\underset{\sim}{z}} \\ \dot{\hat{\underset{\sim}{p}}} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{P}\underset{\sim}{A}\underset{\sim}{P}^{-1} & -\underset{\sim}{P}\underset{\sim}{B}R^{-1}\underset{\sim}{B}^{T}\underset{\sim}{P}^{T} \\ -(\underset{\sim}{P}^{-1})^{T}\,\underset{\sim}{Q}\,\underset{\sim}{P}^{T} & -(\underset{\sim}{P}^{-1})^{T}\underset{\sim}{A}^{T}\underset{\sim}{P}^{T} \end{bmatrix} \begin{bmatrix} \underset{\sim}{z} \\ \hat{\underset{\sim}{p}} \end{bmatrix} \qquad (4\text{-}22)$$

$$\begin{bmatrix} \dot{\underset{\sim}{z}} \\ \dot{\hat{\underset{\sim}{p}}} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{H} & -\underset{\sim}{V}R^{-1}\underset{\sim}{V}^{T} \\ -\tilde{\underset{\sim}{Q}} & -\underset{\sim}{H}^{T} \end{bmatrix} \begin{bmatrix} \underset{\sim}{z} \\ \hat{\underset{\sim}{p}} \end{bmatrix} \qquad (4\text{-}23)$$

Let

$$\underset{\sim}{M} = \begin{bmatrix} \underset{\sim}{A} & -\underset{\sim}{B}R^{-1}\underset{\sim}{B}^{T} \\ -\underset{\sim}{Q} & -\underset{\sim}{A}^{T} \end{bmatrix} ,$$

$$\underset{\sim}{N} = \begin{bmatrix} \underset{\sim}{H} & -\underset{\sim}{V}R^{-1}\underset{\sim}{V}^{T} \\ -\underset{\sim}{Q} & -\underset{\sim}{H}^{T} \end{bmatrix} ,$$

and

$$\tilde{\underset{\sim}{P}} = \begin{bmatrix} \underset{\sim}{P} & \underset{\sim}{0} \\ \underset{\sim}{0} & (\underset{\sim}{P}^{-1})^{T} \end{bmatrix} , \qquad (4\text{-}24)$$

where each sub-matrix of $\underset{\sim}{M}$, $\underset{\sim}{N}$, and $\underset{\sim}{P}$ are known to be (nxn) square matrices. It is easily seen that $\underset{\sim}{\tilde{P}}$ is non-singular, and that

$$\underset{\sim}{\tilde{P}}\;\underset{\sim}{\tilde{P}}^{-1} = \begin{bmatrix} \underset{\sim}{P} & \underset{\sim}{0} \\ \underset{\sim}{0} & (P^{-1})^T \end{bmatrix} \begin{bmatrix} \underset{\sim}{P}^{-1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{P}^T \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{P}\;\underset{\sim}{P}^{-1} & \underset{\sim}{0} \\ 0 & (P^{-1})^T P^T \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{I} \end{bmatrix} = \underset{\sim}{I} \quad . \qquad (4\text{-}25)$$

Therefore,

$$\underset{\sim}{\tilde{P}}\;\underset{\sim}{M}\;\underset{\sim}{\tilde{P}}^{-1} = \underset{\sim}{N} \qquad (4\text{-}26)$$

shows the similarity of the $\underset{\sim}{M}$ and $\underset{\sim}{N}$ matrices. Two similar matrices have the following properties:

1. Their determinants are the same.

$$\det \underset{\sim}{M} = \det \underset{\sim}{N} \qquad (4\text{-}27)$$

2. Their traces are the same.

$$\mathrm{Tr}\,[\underset{\sim}{M}] = \mathrm{Tr}[\underset{\sim}{N}] \tag{4-28}$$

$$\sum_{i=1}^{k} m_{ii} = \sum_{j=1}^{k} n_{jj} \tag{4-29}$$

3. Their characteristic equations are the same.

$$\det[\lambda I-M] = \det[\lambda I-N] = 0 \quad , \tag{4-30}$$

where λ is an arbitrary variable. Since their characteristic equations are the same, the eigenvalues of $\underset{\sim}{M}$ and $\underset{\sim}{N}$ must be identical. It is now known that the Cauer Second Form and phase variable system matrices are similar in that they possess identical eigenvalues.

D. DEVELOPMENT OF THE PRESCRIBED EIGENVALUE PROBLEM

Initially given is a known linear system with dynamics described by either a set of first order differential equations or its transfer function. It is desired to find the optimal feedback control, u*, such that the performance measure

$$J = \tfrac{1}{2}\{ \int_{0}^{\infty} [\, \underset{\sim}{x}^{T} \begin{bmatrix} q_{11} & 0 & \cdots\cdots & 0 \\ 0 & q_{22} & \cdots\cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots\cdots & q_{nn} \end{bmatrix} x \; +$$

$$u^T[1]u] \} \, dt \qquad (4\text{-}31)$$

is minimized, where the eigenvalues of the optimal system are specified as

$$\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \quad .$$

1. Evaluation of the State (H) and Linear Transformation ($\underset{\sim}{P}$) Matrices

Assume the transfer function of the known system is given

$$T(s) = \frac{\sum_{i=1}^{n} \beta_i s^{i-1}}{s^n + \sum_{i=1}^{n} \quad {}_i s^{i-1}} \qquad (4\text{-}32)$$

The system matrices in phase variable form are:

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ -\alpha_1 & -\alpha_2 & -\alpha_3 \cdots\cdots & & -\alpha_n \end{bmatrix} \qquad \underset{\sim}{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \qquad (4\text{-}33)$$

$$\underset{\sim}{C} = [\beta_1 \; \beta_2 \; \ldots . \; \beta_n] \qquad (4\text{-}33)$$

From equation (4-32), the Routh array is formed:

$$\begin{array}{l}
\alpha_1 \quad \alpha_2 \cdots\cdots\cdots\cdots \alpha_n \quad 1 \\
\\
\beta_1 \quad \beta_2 \cdots\cdots\cdots\cdots \beta_n \quad 0 \\
\\
\boxed{C_1 \quad C_2 \cdots\cdots\cdots\cdots 1} \\
\\
d_1 \quad d_2 \cdots\cdots\cdots\cdots 0 \\
\\
\boxed{e_1 \quad e_2 \qquad\qquad 1} \\
\vdots \qquad\qquad\qquad 0 \\
\vdots \\
\boxed{1} \\
\\
0
\end{array} \tag{4-34}$$

In matrix notation, the Routh array becomes $[r_{ij}]$, where

$$i \varepsilon [1,\ 2,\ \ldots.,\ 2(n+1)]$$

$$j \varepsilon [1,\ 2,\ \ldots.,\ n+1] \quad ,$$

and elements

$$r_{2(n-k)+3,\ k} = 1$$

$$r_{2(n-k)-4,\ k} = 0$$

$$k \varepsilon [1,\ 2,\ \ldots.,\ n+1] \ .$$

In general, the $(2L+1)$th row of the Routh array is the Lth row of $\underset{\sim}{P}$.

$$\underset{\sim}{P} = \begin{bmatrix} r_{31} & r_{32} & r_{33} & \cdots\cdots\cdots & r_{3,n} \\ 0 & r_{51} & r_{52} & \cdots\cdots\cdots & r_{5,n-1} \\ 0 & 0 & r_{71} & \cdots\cdots\cdots & r_{7,n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & r_{2n+1,2} \\ 0 & 0 & 0 & \cdots\cdots\cdots & r_{2n+1,1} \end{bmatrix} \tag{4-35}$$

$\underset{\sim}{P}$ being an (nxn) upper triangular matrix, will always have an inverse, $\underset{\sim}{P}^{-1}$, which can be quickly and efficiently determined by digital computer. The $\underset{\sim}{H}$ matrix formulation becomes:

$$\underset{\sim}{H} = \underset{\sim}{P}\ \underset{\sim}{A}\ \underset{\sim}{P}^{-1} \tag{4-36}$$

$$= \begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots\cdots\cdots & h_{1n} \\ h_{11} & h_{22} & h_{23} & \cdots\cdots\cdots & h_{2n} \\ \vdots & h_{22} & h_{33} & \cdots\cdots\cdots & h_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & h_{n-1,n} \\ h_{11} & h_{22} & h_{33} & \cdots\cdots\cdots & h_{n,n} \end{bmatrix} \tag{4-37}$$

The elements of the $\underset{\sim}{H}$ matrix can also be obtained more easily and directly from the first column of the Routh array.

Let

$$h_i = \frac{r_{i,1}}{r_{i+1,1}} \tag{4-38}$$

The h_i's correspond to the quotients of the continued fraction expansion of the transfer function in (32).

$$T(s) = \cfrac{1}{h_1 + \cfrac{s}{h_2 + \cfrac{s}{h_3 + \cfrac{s}{h_4 + \cdot \; \cdot \; \cdot}}}} \tag{4-39}$$

The $\underset{\sim}{H}$ matrix then becomes as shown in Figure 4.1. The regular pattern of the elements enable the $\underset{\sim}{H}$ matrix to be obtained by inspection once the h_i's have been determined from either (4-34) and (4-38), or (4-39).

The matrices $\underset{\sim}{V}$, and $\underset{\sim}{C}^*$, are easily obtained:

$$\underset{\sim}{V} = \underset{\sim}{P}\,\underset{\sim}{B} = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} . \tag{4-40}$$

$$\underset{\sim}{H} = - \begin{bmatrix} h_2h_1 & h_4h_1 & h_6h_1 & \cdots\cdots\cdots & h_{2n}h_1 \\ h_2h_1 & h_4(h_1+h_3) & h_6(h_1+h_3) & \cdots\cdots & h_{2n}(h_1+h_3) \\ h_2h_1 & h_4(h_1+h_3) & h_6(h_1+h_3+h_5) & \cdots\cdots & h_{2n}(h_1+h_3+h_5) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_2h_1 & h_4(h_1+h_3) & h_6(h_1+h_3+h_5) & \cdots & h_{2n}(h_1+\cdot\cdot+h_{2n-1}) \end{bmatrix}$$

Figure 4.1. State Matrix, $\underset{\sim}{H}$, in Cauer Second Form

$$\underset{\sim}{C}^{*} = \underset{\sim}{C}\,\underset{\sim}{P}^{-1} = [h_2 \quad h_4 \quad h_6 \; \ldots\ldots \; h_{2n}] \tag{4-41}$$

B. Evaluation of $\tilde{\underset{\sim}{Q}}$

$$\tilde{\underset{\sim}{Q}} = (\underset{\sim}{P}^{-1})^{T}\,\underset{\sim}{Q}\,\underset{\sim}{P}^{-1}$$

$$= \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} & \tilde{Q}_{13} & \cdots\cdots\cdots & \tilde{Q}_{1n} \\ \tilde{Q}_{12} & \tilde{Q}_{22} & \tilde{Q}_{23} & \cdots\cdots\cdots & \tilde{Q}_{2n} \\ \tilde{Q}_{13} & \tilde{Q}_{23} & \tilde{Q}_{33} & \cdots\cdots\cdots & \tilde{Q}_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ \tilde{Q}_{n1} & \tilde{Q}_{n2} & \tilde{Q}_{n3} & \cdots\cdots\cdots & \tilde{Q}_{nn} \end{bmatrix} \tag{4-42}$$

The canonical system in Cauer Second Form (4-23) has now been obtained, with the numerical values of the elements of $\tilde{\underset{\sim}{Q}}$ still to be found. This system will be compared to a non-optimized system with the prescribed eigenvalues also in Cauer Second Form. The desired system in phase variable form is:

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}^{*}\underset{\sim}{x} + \underset{\sim}{B}^{*}u \tag{4-43}$$

$$y = \underset{\sim}{E}\,\underset{\sim}{x} \quad . \tag{4-44}$$

Formulation into Cauer Second Form yields

$$\dot{\underset{\sim}{x}} = \underset{\sim}{H}^{*}\underset{\sim}{z} + \underset{\sim}{V}^{*}u \tag{4-45}$$

$$y = \underset{\sim}{E}^{*}\underset{\sim}{z} \quad . \tag{4-46}$$

In a "nonoptimized" system ($\underset{\sim}{Q}$ and $\underset{\sim}{R}$ set equal to $\underset{\sim}{0}$), the canonical system appears as:

$$\begin{bmatrix} \dot{\underset{\sim}{z}} \\ \dot{\underset{\sim}{p}} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{H}^* & 0 \\ 0 & -(\underset{\sim}{H}^*)^T \end{bmatrix} \begin{bmatrix} \underset{\sim}{z} \\ \underset{\sim}{p} \end{bmatrix} , \qquad (4\text{-}47)$$

where

$$\begin{bmatrix} \underset{\sim}{H}^* & \underset{\sim}{0} \\ 0 & -(H^*)^T \end{bmatrix} \equiv \underset{\sim}{N}^* \qquad (4\text{-}48)$$

possesses the n prescribed eigenvalues with negative real parts and n eigenvalues with positive real parts, symmetric about the imaginary axis in the complex plane.

By equating the characteristic polynomials of $\underset{\sim}{N}$ and $\underset{\sim}{N}^*$, (4-23) and (4-47) respectively, it is now possible to determine the elements of the $\underset{\sim}{\tilde{Q}}$ weighting matrix.

$$\det[s\underset{\sim}{I}-\underset{\sim}{N}] = \det[s\underset{\sim}{I}-\underset{\sim}{N}^*] \qquad (4\text{-}49)$$

$$\det \begin{bmatrix} s\underset{\sim}{I}-\underset{\sim}{H} & \underset{\sim}{V}R^{-1}\underset{\sim}{V}^T \\ \underset{\sim}{\tilde{Q}} & s\underset{\sim}{I}+\underset{\sim}{H}^T \end{bmatrix} \doteq$$

$$\det \begin{bmatrix} s\underset{\sim}{I}-\underset{\sim}{H}^* & \underset{\sim}{0} \\ \underset{\sim}{0} & s\underset{\sim}{I}+(\underset{\sim}{H}^*)^T \end{bmatrix} \qquad (4\text{-}50)$$

E. DETERMINATION OF THE OPTIMAL CONTROL LAW

Finding the elements of the weighting matrix, $\tilde{\underset{\sim}{Q}}$, is obviously a non-trivial matter for all but the lowest order systems. The method developed for obtaining these values is based upon a succession of matrix building blocks, which are individually computationally simple.

Starting with the matrix, $\underset{\sim}{H}$,

$$\underset{\sim}{H} = [h_{ij}] \tag{4-51}$$

define a new matrix, $\underset{\sim}{T}$,

$$\underset{\sim}{T} = [t_{ij}] \tag{4-52}$$

where

$$t_{ij} = h_{ij} - h_{jj} \qquad i \neq j \tag{4-53}$$

and

$$t_{ii} = \sum_{j=i+1}^{n} t_{ij} \quad . \qquad i \neq n \tag{4-54}$$

The matrix, $\underset{\sim}{T}$, therefore, is an (nxn) upper right triangular matrix, where the diagonal elements are equal to the sum of all other elements in the same row. The next "building block" matrix, $\underset{\sim}{G}$, is defined by:

$$G = [g_{ij}] \tag{4-55}$$

$$g_{i,1} = 1.0 \tag{4-56}$$

for $i \varepsilon [1, 2, \ldots, n]$,

$$g_{i,2} = t_{j-2,j-1} \tag{4-57}$$

for $j \varepsilon [1, 2, \ldots, n-1]$,

$$g_{ij} = \sum_{k=i+1}^{n-j+2} (t_{i,k} \times g_{k,j-1}) \tag{4-58}$$

for $j \varepsilon [3, 4, \ldots, x]$ and $i E [1, 2, \ldots, n-j+1]$, where the index j is held fixed for each summation over the index i. The matrix, $\underset{\sim}{G}$, is an (nxn) upper left triangular matrix characterized by the first column being all ones. One more matrix needs to be defined at this point. Let the matrix, $\underset{\sim}{W}$, be defined as:

$$\underset{\sim}{W} = [\underset{-}{w}_{ijk}] \tag{4-59}$$

where

$$\underset{-}{w}_{i,j,k} = (-1)^j \times g_{i,j} \tag{4-60}$$

for i, j and k $\underline{\varepsilon}[1, 2, \ldots, n]$.

The matrix, $\underset{\sim}{W}$, is therefore a tridimensional array with each "level" an upper left triangular matrix. Examples of the matrices, $\underset{\sim}{T}$, $\underset{\sim}{G}$ and $\underset{\sim}{W}$ are shown below. Although not evident at this point, the G and W matrices will be used heavily in obtaining the values of the elements of the $\tilde{Q}$ matrix.

From the linear transformation

$$\underset{\sim}{\tilde{Q}} = (\underset{\sim}{P}^{-1})^T \underset{\sim}{Q} \underset{\sim}{P}^T \quad , \tag{4-61}$$

it is observed that once the element $\underset{\sim}{Q}_{ii}$ is known, the remaining elements in the same row (column) can then be obtained through a process of

$$\underset{\sim}{T} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & \cdots\cdots & t_{1n} \\ 0 & t_{22} & t_{23} & \cdots\cdots & t_{2n} \\ \vdots & 0 & t_{33} & \cdots\cdots & t_{3n} \\ \vdots & \vdots & 0 & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots\cdots & t_{nn} \end{bmatrix} \tag{4-62}$$

$$\underset{\sim}{G} = \begin{bmatrix} 1 & t_{11} & g_{13} & \cdots\cdots g_{1,n-1} & g_{1,n} \\ 1 & t_{22} & g_{23} & \cdots\cdots g_{2,n-1} & 0 \\ 1 & t_{33} & g_{33} & \cdots\cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & g_{n-2,3} & \vdots & \vdots \\ 1 & t_{n-1,n-1} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4-63}$$

$$\underset{\sim}{W} = \begin{bmatrix} -1 & \sum_{j=2}^{n} t_{ij} & -\sum_{j=i}^{n-1} t_{ij}xg_{j+1,2} & \sum_{j=2}^{n-2} t_{ij}xg_{j+1,3} & \cdots & \sum_{j=2}^{2} t_{ij}xg_{j+1,n-1} \\ -1 & \sum_{j=3}^{n} t_{2j} & -{}_{j=3}^{n-1} t_{2j}xg_{j+1,2} & {}_{j=3}^{n-2} t_{2j}xg_{j+1,3} & \cdots & 0 \\ -1 & \sum_{j=4}^{n} t_{3j} & -\sum_{j=4}^{n-1} t_{3j}xg_{j+1,2} & \sum_{j=4}^{n-2} t_{3j}xg_{j+1,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & \sum_{j=n-2}^{n-2} t_{n-2,j}xg_{j+1,3} & & \vdots \\ \vdots & \vdots & -\sum_{j=n-1}^{n-1} t_{n-1,n-1}xg_{j+1,2} & 0 & \cdots\cdots\cdots & 0 \\ -1 & \sum_{j=n}^{n} t_{n-1,j} & 0 & 0 & \cdots\cdots\cdots & 0 \\ -1 & 0 & 0 & 0 & \cdots\cdots\cdots & 0 \end{bmatrix} \qquad (4\text{-}64)$$

linear combinations of previously determined values. The $\tilde{Q}$ matrix is thus found in the following manner:

$$\begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} & \tilde{Q}_{13} & \tilde{Q}_{14} & \cdots & \tilde{Q}_{1n} \\ & \tilde{Q}_{22} & \tilde{Q}_{23} & \tilde{Q}_{24} & \cdots & \tilde{Q}_{2n} \\ & & \tilde{Q}_{33} & \tilde{Q}_{34} & \cdots & \tilde{Q}_{3n} \\ & & & \tilde{Q}_{44} & \cdots & \tilde{Q}_{4n} \\ & & & & & \\ & & & & & \tilde{Q}_{nn} \end{bmatrix} \tag{4-65}$$

where for an nth order system ($n \geq 1$)

$$\tilde{Q}_{11} = \frac{\prod_{i=1}^{2n} (H_i)^2 - \prod_{i=1}^{2n} (h_i)^2}{\prod_{i=3}^{2n} (h_i)^2}$$

$$= \prod_{i=3}^{2n} \left(\frac{H_i}{h_i}\right)^2 \left[(H_2H_1)^2 - (h_2h_1)^2\right]^{+} \tag{4-66}$$

[+] for n=1, $\tilde{Q}_{11} = (H_2H_1)^2 - (h_2h_1)^2$.

$$\tilde{Q}_{ij} = - \sum_{k=1}^{j-1} \frac{P_{k,n-1}}{P_{n-1,n-1}} \tilde{Q}_{ik} \qquad (4\text{-}67)$$ [+]

for $i \neq j$ or $j \neq n$,

$$\tilde{Q}_{in} = - \sum_{j=1}^{n-1} \tilde{Q}_{ij} \qquad (4\text{-}67)$$

and

$$\tilde{Q}_{nn} = - \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{Q}_{ij} + \left(\sum_{i=1}^{n} H_{ii} \right)^2 - \left(\sum_{j=1}^{n} h_{jj} \right)^2$$

$$+ 2\left[\sum_{i=1}^{n} \sum_{j=1}^{n} H_{ii}(H_{jj}-H_{ij}) - \sum_{L=1}^{n} \sum_{j=1}^{n} h_{ij}(h_{jj}-h_{ij}) \right] \qquad (4\text{-}69)$$

The values of the diagonal elements of $\tilde{Q}$ (other than $\tilde{Q}_{11}$) generally involve varying linear combinations of already determined values, more easily expressed in terms of the $\underset{\sim}{G}$ and $\underset{\sim}{W}$ matrices, rather than the $\underset{\sim}{P}$ and $\underset{\sim}{H}$ matrices. To aid in determining the values along the diagonal the following labels are provided for $\underset{\sim}{G}$ and $\underset{\sim}{W}$.

[+] $P_{i,j} = \underset{\sim}{P}$

$$\begin{array}{cc} & \begin{array}{cccccc} s^{N-1} & s^{N-2} & s^{N-3} & & s^{1} & s^{0} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ \\ \end{array} & \left[\begin{array}{cccccc} g_{11} & g_{12} & g_{13} & \cdots & g_{1,n-1} & g_{1n} \\ g_{21} & g_{22} & g_{23} & \cdots & g_{2,n-1} & 0 \\ g_{31} & g_{32} & g_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ g_{n-1,1} & g_{n-1,2} & & & \vdots & \vdots \\ g_{n1} & 0 & & & 0 & 0 \end{array}\right] \end{array} \qquad (4\text{-}70)$$

$$\begin{array}{cc} & \begin{array}{cccccc} s^{N-1} & s^{N-2} & s^{N-3} & & & s^{0} \\ W_{11n} & W_{12n} & W_{13n} & \cdots & \cdots & W_{1nn} \end{array} \\ \begin{array}{c} \tilde{Q}_{i1} \\ \tilde{Q}_{i2} \\ \tilde{Q}_{i3} \\ \vdots \\ \tilde{Q}_{in-1} \\ \tilde{Q}_{i,n} \end{array} & \left[\begin{array}{cccccc} W_{11i} & W_{12i} & W_{13i} & \cdots & W_{1,n-1,i} & W_{1,n,i} \\ W_{21j} & W_{22i} & W_{23}{}^{i} & \cdots & W_{2,n-1,i} & 0 \\ W_{31i} & W_{32i} & W_{33i} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ W_{n-1,i,j} & W_{n-1,2,i} & 0 & \cdots & 0 & 0 \\ W_{n,1,i} & 0 & 0 & \cdots & 0 & 0 \end{array}\right] \end{array}$$

(4-71)

These two arrays will provide the coefficients of each "$Q_{ij} \times s^k$" necessary. Assume, for example, that a 5th order system is being considered in equation (4-50). The results of the expansion of both sides of the equation results in:

$$a_o + \sum_{i=1}^{n} a_i s^{2i} = b_o + \sum_{L=1}^{n} b_i s^{2i} \tag{4-72}$$

where $a_n = 1$, $b_n = 1$. By equating a_{i-1} and b_{i-1} for $i \varepsilon [1, 2, \ldots, n]$, it is possible to obtain $\tilde{Q}_{ii}$.

i.e. $$a_o = b_o \tag{4-73}$$

a_o and b_o are the coefficients of s^0. From (4-70), it is observed that the only element in the s^0 column that is non-zero is g_{15}, which appears in row 1. The 1 "indicates" that it is necessary to only look in "level" 1 of $\tilde{W}$, in the column corresponding to s^0. This yields only the single element W_{151}. Therefore, the coefficient of $Q_{11}s^0$ will be

$$g_{15} \times W_{151} \quad . \tag{4-74}$$

What remains to be determined are the coefficients of s^{2i} not involving the $\tilde{Q}_{ij}$'s. Because of the symmetry of the eigenvalues of both the system being "optimized" and the system with the prescribed eigenvalues, the remaining coefficients (those not involving a $\tilde{Q}_{ij}$) are easily determined from:

$$\det[sI-H] \times \det[sI\_H^{T}] \qquad (4\text{-}75)$$

and

$$\det[sI-H^{*}] \times \det[sI+(H^{*})^{T}], \qquad (4\text{-}76)$$

which give:

$$\left(\sum_{i=0}^{n} \alpha_i s^i \right)\left(\sum_{i=0}^{n} (-1)^k \alpha_i s^i \right) \qquad (4\text{-}77)$$

and

$$\left(\sum_{i=0}^{n} \alpha_i^{*} s^i \right)\left(\sum_{i=0}^{n} (-1)^k \alpha_i^{*} s^i \right) \qquad (4\text{-}78)$$

where k=i+1 for n even and k=i for n odd. The indicated multiplication in (4-77) and (4-78) result in:

$$\sum_{i=0}^{n} (-1)^k \sigma_i s^{2i} \qquad (4\text{-}79)$$

and

$$\sum_{i=0}^{n} (-1)^k \sigma_i^{*} s^{2i} \qquad (4\text{-}80)$$

respectively, where the same conditions are imposed on "k".

Returning to the 5th order example, the coefficients of s^0 are $-\sigma_0$ and $-\sigma_0^{*}$. To obtain $\tilde{Q}_{11}$ is now a matter of solving the equation:

$$(g_{15} \cdot W_{151})\, \tilde{Q}_{11} - \sigma_0 = -\sigma_0^{*} \qquad (4\text{-}81)$$

$$Q_{11} = \frac{\sigma_0 - \sigma_0^*}{(g_{15} \cdot g_{151})} \tag{4-82}$$

With $\tilde{Q}_{11}$ known, it is a simple matter to obtain the remaining elements in the first row (and column) using (4-67) and (4-68).

To find $\tilde{Q}_{22}$, find all the coefficients of s^2. s^2 results in three separate ways:

$$\begin{aligned} &(1) \quad s^0 \times s^2 \\ &(2) \quad s^1 \times s^1 \\ &(3) \quad s^2 \times s^0 \quad . \end{aligned} \tag{4-83}$$

The multiplicand indicates which columns of $\underset{\sim}{G}$, (4-70), is of interest. Any non-zero element in $\underset{\sim}{G}$ tells which level of $\underset{\sim}{W}$ is of interest. The multiplier is the indicator for the column of interest in the array, $\underset{\sim}{W}$. Therefore, for $s^0 \times s^2$:

$$g_{15} \times (W_{131}\ \tilde{Q}_{11} + W_{231}\ \tilde{Q}_{12} + W_{331}\ \tilde{Q}_{13}) \tag{4-84}$$

for $s^1 \times s^1$:

$$\begin{aligned} &g_{1,4} \times (W_{141}\ \tilde{Q}_{11} + W_{241}\ \tilde{Q}_{12}) + \\ &g_{2,4} \times (W_{142}\ \tilde{Q}_{21} + W_{242}\ \tilde{Q}_{22}) \end{aligned} \tag{4-85}$$

for $s^2 \times s^0$:

$$g_{1,3} \times (W_{151} \tilde{Q}_{11}) +$$

$$g_{2,3} \times (W_{152} \tilde{Q}_{21}) +$$

$$g_{3,3} \times (W_{153} \tilde{Q}_{31}) \quad . \tag{4-86}$$

Obtaining $\tilde{Q}_{22}$ is now a matter of solving the equation given by the combination of (4-79), (4-80), (-84), (4-85) and (4-86):

$$g_{15} \left(\sum_{i=1}^{3} W_{i31} \tilde{Q}_{1i}\right) + \left(\sum_{j=i}^{2} g_{j,4} \sum_{i=1}^{2} W_{i4j} \tilde{Q}_{ji}\right)$$

$$\left(\sum_{j=1}^{3} g_{j,3} W_{15j} \tilde{Q}_{ji}\right) - \sigma_1 = -\sigma_1{}^* \; , \tag{4-87}$$

for $\tilde{Q}_{22}$.

Once $\tilde{Q}_{22}$ has been found, the remainder of the 2nd row (and column) can be obtained using (4-67) and (4-68).

In a like manner, the respective equations to be solved for $\tilde{Q}_{33}$, $\tilde{Q}_{44}$ and $\tilde{Q}_{55}$ are:

$$\left(\sum_{j=1}^{1} g_{j,5} \sum_{i=1}^{5} W_{i1j} \tilde{Q}_{ji}\right) + \left(\sum_{j=1}^{2} g_{j,4} \sum_{i=1}^{4} W_{i2j} \tilde{Q}_{ji}\right)$$

$$\underline{+ \left(\sum_{j=1}^{3} g_{j,3} \sum_{i=1}^{3} W_{i3j} \tilde{Q}_{ji}\right)} + \left(\sum_{j=1}^{4} g_{j,2} \sum_{i=1}^{2} W_{i4j} \tilde{Q}_{ji}\right)$$

$$+ \left(\sum_{j=1}^{5} g_{j,1} \sum_{i=1}^{1} W_{i5j} \tilde{Q}_{ji}\right) - \sigma_2 = -\sigma_2{}^* \; , \tag{4-88}$$

$$\left(\sum_{j=1}^{3} g_{j,3} \sum_{i=1}^{5} W_{i1j} \tilde{Q}_{ji}\right) + \underline{\left(\sum_{i=1}^{4} g_{j,2} \sum_{i=1}^{4} W_{i2j} \tilde{Q}_{ji}\right)}$$

$$+ \left(\sum_{j=1}^{5} g_{j,1} \sum_{i=1}^{3} W_{i3j} \tilde{Q}_{ji}\right) + \sigma_3 = \sigma_3^{*} , \qquad (4\text{-}89)$$

$$\underline{\left(\sum_{j=1}^{5} g_{j,1} \sum_{i=1}^{5} W_{i1j} \tilde{Q}_{ji}\right)} - \sigma_4 = -\sigma_4^{*} . \qquad (4\text{-}90)$$

The underscore in (4-88), (4-89) and (4-90) indicates the term that contains $\tilde{Q}_{33}$, $\tilde{Q}_{44}$, and $\tilde{Q}_{55}$, respectively.

The entire $\underset{\sim}{\tilde{Q}}$ matrix is now known. By the inverse transformation of (4-42),

$$\underset{\sim}{Q} = \underset{\sim}{P}^{T} \underset{\sim}{\tilde{Q}} \underset{\sim}{P} . \qquad (4\text{-}91)$$

$\underset{\sim}{Q}$ along with $\underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{C}$ of (4-1) and (4-2) are used to obtain the matrix Riccati equation steady state solution.

$$\underset{\sim}{0} = \underset{\sim}{K}\underset{\sim}{A} + \underset{\sim}{A}^{T}\underset{\sim}{K} - \underset{\sim}{K}\underset{\sim}{B}R^{-1}\underset{\sim}{B}^{T}\underset{\sim}{K} + \underset{\sim}{Q} . \qquad (4\text{-}92)$$

Once $\underset{\sim}{K}$ has been determined, the optimal control law is given by

$$\underset{\sim}{u}^{*} = -R^{-1}\underset{\sim}{B}\underset{\sim}{K}\underset{\sim}{x} .^{+} \qquad (4\text{-}93)$$

Once u* has been determined, an inverse non-singular transformation can be performed to take the phase variable form

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[+] It is now known that $\underset{\sim}{G} = -R^{-1}\underset{\sim}{B}^{T}\underset{\sim}{K}$ in (4-20).

back into state variable form. (See Appendix A)

F. ILLUSTRATE EXAMPLES

1. Odd Order System (Third)

Given the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -13 & -19 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad (4\text{-}94)$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} , \qquad (4\text{-}95)$$

find the optimal control law, u^*, such that the quadratic performance index (4-31) is minimized, where the eigenvalues of the system are specified as

$s_1 = -3$, $s_2 = -4$, $s_3 = -6$.

Forming the Routh array yields

$$\begin{array}{llll}
13 & 19 & 7 & 1 \\
1 & 0 & 0 & 0 \\
19 & 7 & 1 & \\
-7/19 & -1/19 & 0 & \\
30/7 & 1 & & \\
1/30 & 0 & & \\
1 & & & \\
0 & & &
\end{array}$$

from which the third, fifth and seventh row are extracted to form $\underset{\sim}{P}$.

$$\underset{\sim}{P} = \begin{bmatrix} 19 & 7 & 1 \\ 0 & 30/7 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4-97}$$

and

$$\underset{\sim}{P}^{-1} = \begin{bmatrix} 1/19 & -49/570 & 1/30 \\ 0 & 7/30 & -7/30 \\ 0 & 0 & 1 \end{bmatrix} \tag{4-98}$$

From (4-97) and (4-98), $\underset{\sim}{H}$ and $\underset{\sim}{V}$ are calculated

$$\underset{\sim}{H}=\underset{\sim}{P}\underset{\sim}{A}\underset{\sim}{P}^{-1}= \begin{bmatrix} -13/19 & 637/570 & -13/30 \\ -13/19 & -63/19 & 9/7 \\ -13/19 & -63/19 & -3 \end{bmatrix} \tag{4-99}$$

$$\underset{\sim}{V} = \underset{\sim}{P}\underset{\sim}{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{4-100}$$

The set of prescribed eigenvalues yields the system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -72 & -54 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{4-101}$$

and

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{4-102}$$

From the Routh array

$$\begin{array}{llll} 72 & 54 & 13 & 1 \\ 1 & 0 & 0 & 0 \\ 54 & 13 & 1 & \\ -13/54 & -1/54 & 0 & \\ 115/13 & 1 & & \\ 1/115 & 0 & & \\ 1 & & & \\ 0 & & & \end{array} \tag{4-103}$$

extracting rows 3, 5 and 7 yields the P* matrix:

$$\underset{\sim}{P}^* = \begin{bmatrix} 54 & 13 & 1 \\ 0 & 115/13 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad (\underset{\sim}{P}^*)^{-1} = \begin{bmatrix} 1/54 & -169/6210 & 1/115 \\ 0 & 13/115 & -13/115 \\ 0 & 0 & 1 \end{bmatrix} \tag{4-104}$$

from which $\underset{\sim}{H}^*$ and $\underset{\sim}{V}^*$ are calculated,

$$\underset{\sim}{H}^* = \underset{\sim}{P}^* \underset{\sim}{A}^* (\underset{\sim}{P}^*)^{-1} = \begin{bmatrix} -4/3 & 676/345 & -72/115 \\ -4/3 & -2574/621 & 1980/1495 \\ -4/3 & -2574/621 & -11254/1495 \end{bmatrix} \tag{4-105}$$

$$\underset{\sim}{V}^* = \underset{\sim}{P}^* \underset{\sim}{B}^* = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{4-106}$$

Formulation of equation (4-50) yields

$$\det \begin{bmatrix} s\underset{\sim}{I} - \underset{\sim}{H} & \underset{\sim}{V} R^{-1} \underset{\sim}{V}^T \\ \underset{\sim}{Q} & s\underset{\sim}{I} + \underset{\sim}{H}^T \end{bmatrix} = \det \begin{bmatrix} s\underset{\sim}{I} - \underset{\sim}{H}^* & \underset{\sim}{0} \\ \underset{\sim}{0} & s\mathrm{I} + \underset{\sim}{H}^T \end{bmatrix}$$

$$\det\begin{bmatrix} s+13/19 & -637/570 & 13/30 & 1 & 1 & 1 \\ 13/19 & s+63/19 & -9/7 & 1 & 1 & 1 \\ 13/19 & 63/19 & s+3 & 1 & 1 & 1 \\ \underset{\sim}{Q}_{11} & \underset{\sim}{Q}_{12} & \underset{\sim}{Q}_{13} & s-13/19 & -13/19 & -13/19 \\ \underset{\sim}{Q}_{12} & \underset{\sim}{Q}_{22} & \underset{\sim}{Q}_{23} & 637/570 & s-63/19 & -63/19 \\ \underset{\sim}{Q}_{13} & \underset{\sim}{Q}_{23} & \underset{\sim}{Q}_{33} & -13/30 & 9/7 & s-3 \end{bmatrix}$$

$$=\det\begin{bmatrix} s+4/3 & -676/345 & 72/115 & 0 & 0 & 0 \\ 4/3 & s+2574/621 & -1980/1495 & 0 & 0 & 0 \\ 4/3 & 2574/621 & s+11245/1495 & 0 & 0 & 0 \\ 0 & 0 & 0 & s-4/3 & -4/3 & -4/3 \\ 0 & 0 & 0 & 676/345 & s-2574/621 & -2574/621 \\ 0 & 0 & 0 & -72/115 & 1980/1495 & s-11245/1495 \end{bmatrix} \qquad (4\text{-}107)$$

It is desired to determine the values of $\underset{\sim}{Q}$. Brute force enumeration of the determinants and equating coefficients of like powers of s would eventually lead to the solution. Using instead the method developed, from $\underset{\sim}{H}$(not H*) evolve the $\underset{\sim}{T}$, $\underset{\sim}{G}$, and $\underset{\sim}{W}$ matrices.

$$\underset{\sim}{T} = \begin{bmatrix} 7.0000 & 4.4333 & 2.5667 \\ 0 & 4.2857 & 4.2857 \\ 0 & 0 & 0 \end{bmatrix} \qquad (4\text{-}108)$$

$$\underset{\sim}{G} = \begin{bmatrix} 1 & 7.0000 & 19.0000 \\ 1 & 4.2857 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{4-109}$$

$$\text{and } \underset{\sim}{W} = \begin{bmatrix} -1 & 7.0000 & -19.0000 \\ -1 & 4.2857 & 0 \\ -1 & 0 & 0 \end{bmatrix} \tag{4-110}$$

Now add the appropriate labels as in (4-70) and (4-71)

for $\underset{\sim}{G}$:

$$\begin{array}{c|ccc|} & s^2 & s^1 & s^0 \\ 1 & 1 & 7.0 & 19.0 \\ 2 & 1 & 4.2857 & 0 \\ 3 & 1 & 0 & 0 \end{array} \tag{4-111}$$

for $\underset{\sim}{W}$:

$$\begin{array}{c|ccc|} & s^2 & s^1 & s^0 \\ \underset{\sim}{Q}_{i1} & -1 & 7.0 & -19.0 \\ \underset{\sim}{Q}_{i2} & -1 & 4.2857 & 0 \\ \underset{\sim}{Q}_{i3} & -1 & 0 & 0 \end{array} \tag{4-112}$$

Still necessary are the coefficients in (4-79) and (4-80):

$$\left(\sum_{i=0}^{n} \alpha_i s^i\right)\left(\sum_{i=0}^{n} (-1)^k \alpha_i s^i\right) = \sum_{i=0}^{n} (-1)^k \sigma_i s^{2i} \tag{4-113}$$

for n odd (n=3), k=i+1,

$$\left(\sum_{i=0}^{n} \alpha_i s^i\right)\left(\sum_{i=0}^{n} (-1)^{i+1} \alpha_i s^i\right) = \sum_{i=0}^{n} (-1)^{i+1} \alpha_i s^{2i} \qquad (4\text{-}114)$$

$$(s^3+7s^2+19s+13)(s^3-7s^2+19s-13)$$

$$= s^6 - 11s^4 + 179s^2 - 169, \qquad (4\text{-}115)$$

and

$$\left(\sum_{i=0}^{3} \alpha_i^* s^i\right)\left(\sum_{i=0}^{3} (-1)^{i+1} \alpha_i^* s^i\right) = \sum_{i=0}^{3} (-1)^{i+1} \alpha_i^* s^{2i} \qquad (4\text{-}116)$$

$$(s^3+13s^2+54s+72)(s^3-13s^2+54s-72)$$

$$= s^6 - 61s^4 + 1044s^2 - 5184 . \qquad (4\text{-}117)$$

Now, by equating coefficients of s^0, it is possible to obtain $\tilde{Q}_{11}$. From $\underset{\sim}{G}$, the only non-zero element in the "s^0 column" is 19, which corresponds to row 1, and therefore level 1 of $\underset{\sim}{W}$. From level 1 of $\underset{\sim}{W}$, the only non-zero element in the "s^0 column" is -19, with row label Q_{11}. The coefficient of $\tilde{Q}_{11}$ as obtained from (4-81) is 19(-19) = -361. The solution for $\tilde{Q}_{11}$ is obtained from equations (4-81) and (4-82):

$$-361\ \tilde{Q}_{11} - 169 = -5184$$

$$\tilde{Q}_{11} = (-5184 + 169)\ /\ -361$$

$$\cong 13.892\ . \tag{4-118}$$

From (4-65), (4-67) and (4-68), $\tilde{Q}_{12}$ and $\tilde{Q}_{13}$ are successively obtained:

$$\tilde{Q}_{12} = \frac{r_{12}}{r_{22}}\ \tilde{Q}_{11} = -\frac{1}{30/7}\ (13.892) \cong -22.69 \tag{4-119}$$

$$\tilde{Q}_{13} = -\tilde{Q}_{11} - \tilde{Q}_{12} = -13.892 - (-22.69) \cong 8.798. \tag{4-120}$$

The next power of s which results from expansion of (4-107) is s^2. Equating coefficients of s^2, it is now possible to obtain $\tilde{Q}_{22}$. s^2 results from the products $s^0 x s^2$, $s^1 x s^1$, and $s^2 x s^0$. Therefore, from the development starting at (4-83), the equation to be solved for $\tilde{Q}_{22}$ is:

$$\left(\sum_{j=1}^{1} g_{j,3} \sum_{i=1}^{3} W_{i1j}\ \tilde{Q}_{ji} \right) + \left(\sum_{j=1}^{2} g_{j,2} \sum_{i=1}^{2} W_{i2j}\ \tilde{Q}_{ji} \right)$$

$$+ \left(\sum_{j=1}^{3} g_{j,i} \sum_{i=1}^{1} W_{i3j}\ \tilde{Q}_{ji} \right) + \sigma_1 = \sigma_1{}^{*} \tag{4-121}$$

Substituting known values, equation (4-121) becomes:

$$g_{13}(W_{111}\tilde{Q}_{11} + W_{211}\,\tilde{Q}_{12} + W_{311}\,\tilde{Q}_{13}) +$$

$$g_{12}(W_{121}\,\tilde{Q}_{11} + W_{221}\,\tilde{Q}_{12}) + g_{22}(W_{122}\,\tilde{Q}_{21} + W_{222}\,\tilde{Q}_{22})$$

$$+ g_{11}(W_{131}\,\tilde{Q}_{11}) + g_{21}(W_{132}\,\tilde{Q}_{21}) + g_{31}(W_{133}\,\tilde{Q}_{31})$$

$$+ \sigma_1 = \sigma_1{}^* \qquad (4\text{-}122)$$

$$19.0[-1(13.892) = 1(-22.69) - 1(8.798)]$$

$$+ 7.0\ [7.0(13.892) + 4.2857(-22.69)]$$

$$+ 4.2857\ [7.0(-22.69) + 4.2857\ \tilde{Q}_{22})] + 1[-19(13.892)]$$

$$+ 1\ [-19(-22.69)] + 1[-19(8.798)] + 179 = 1044$$

$$\tilde{Q}_{22} = 84.155 \qquad (4\text{-}123)$$

From (4-68)

$$\tilde{Q}_{23} = -\tilde{Q}_{21} - \tilde{Q}_{22} = -\tilde{Q}_{12} - \tilde{Q}_{22} =$$

$$-\ (-22.69) - (84.155) \cong -61.465 \qquad (4\text{-}124)$$

s^4 is the next power of s obtained in the expansion of (4-107). Equating coefficients of s^4, it is now possible to obtain the equation from which $\tilde{Q}_{33}$ can be found. s^4 results from multiplying $s^2 x s^2$. Therefore, from $\underset{\sim}{G}$ and $\underset{\sim}{W}$:

$$g_{11}[-1(\tilde{Q}_{11}) - 1(\tilde{Q}_{12}) - 1(\tilde{Q}_{13})]$$

$$+ g_{21}[-1(\tilde{Q}_{21}) - 1(\tilde{Q}_{22}) - 1(\tilde{Q}_{23})]$$

$$+ g_{31}[-1(\tilde{Q}_{31}) - 1(\tilde{Q}_{32}) - 1(\tilde{Q}_{33})$$

$$- \sigma_2 = -\sigma_2{}^* \quad , \qquad (4\text{-}125)$$

where g_{11}, g_{21} and g_{31} all equal one. Therefore, (4-125) becomes

$$- \sum_{i=1}^{3} \sum_{j=1}^{3} Q_{ij} - \sigma_2 = -\sigma_2{}^* \qquad (4\text{-}126)$$

Solving (4-126) for $\tilde{Q}_{33}$:

$$\tilde{Q}_{33} = -\sigma_2{}^* + \sigma_2 - \sum_{\substack{i=1 \\ i+j \neq 6}}^{3} \sum_{j=1}^{3} \tilde{Q}_{ij} \quad . \qquad (4\text{-}127)$$

Since

$$\tilde{Q}_{in} = - \sum_{k=1}^{n-1} \tilde{Q}_{ik} \qquad (4\text{-}68)$$

$$\sum_{k=1}^{n} \tilde{Q}_{ik} = 0 \ , \tag{4-128}$$

and (4-127) simplifies to

$$\tilde{Q}_{3k} = + \sigma_2{}^* - \sigma_2 - \sum_{k=1}^{2} \tilde{Q}_{3k} \tag{4-129}$$

$$= 61-11-8.798+61.465 \cong 102.667 \tag{4-130}$$

$\underset{\sim}{\tilde{Q}}$ is now entirely known.

$$\underset{\sim}{\tilde{Q}} = \begin{bmatrix} 13.892 & -22.69 & 8.798 \\ -22.69 & 84.155 & -61.465 \\ 8.798 & -61.465 & 102.667 \end{bmatrix} \tag{4-131}$$

From (4-91):

$$\underset{\sim}{Q} = \underset{\sim}{P}^T \underset{\sim}{\tilde{Q}} \underset{\sim}{P} = \begin{bmatrix} 5015 & 0 & 0 \\ 0 & 865 & 0 \\ 0 & 0 & 50 \end{bmatrix} \tag{4-132}$$

To find the optimal feedback control law, u*, it is necessary to solve the matrix Riccati equation (4-92), where $\underset{\sim}{Q}$, $\underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{C}$ are as given in (4-131), (4-94) and (4-95). The solution yields:

$$u^* = -[59 \quad 35 \quad 6] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{4-133}$$

or $u^* = -59\, x_1 - 35\, x_2 - 6\, x_3$.

With G^* as given in (4-133), $\dot{x} = (A + BG^*)x$ becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -72 & -54 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{4-134}$$

Comparing (4-134) with (4-101), it is observed that $(A+BG^*) \equiv A^*$; therefore, the desired system has been realized with the set of prescribed eigenvalues.

2. Even Order Example (Fourth)

Consider the linear system represented by the transfer function:

$$T_4(s) = \frac{s^2+9s+34}{s^4+12s^3+48s^2+80s+48} \;.^{+} \tag{4.135}$$

[+] the system eigenvalues are -2,-2,-2, and -6.

This system could be expressed in phase variable form (4-31), thereby obtaining the transformation matrix, $\underset{\sim}{P}$; the system can be realized in Cauer Second Form (4-36). Instead, in this example, it is desired to obtain the quotients which result from partial fraction expansion of the transfer function. In Cauer Second Form, T(s) becomes:

$$T(s) = \frac{34+9s+s^2}{49+80s+48s^2+12s^3+s^4}$$

$$= \cfrac{1}{h_1 + \cfrac{s}{h_2 + \cfrac{s}{h_3 + \cfrac{s}{h_4 + \cfrac{s}{\ddots \; h_5 + \cfrac{s}{h_6 + \cfrac{s}{h_7 + \cfrac{s}{h_8}}}}}}}} \, , \qquad (4\text{-}136)$$

where the quotients, h_i, are:

h_1 = 1.41176
h_2 = .505245
h_3 = -4.6287
h_4 = -.627918
h_5 = 23.07
h_6 = .146914
h_7 = -281.808
h_8 = -.024241

Substituting these values into the matrix in Figure 4.1 yields

$$\underset{\sim}{H} = -\begin{bmatrix} .71328 & -.88647 & .20741 & -.03422 \\ .71328 & 2.01997 & -.47261 & .07798 \\ .71328 & 2.01997 & 2.91670 & -.48125 \\ .71328 & 2.01997 & 2.91670 & 6.35004 \end{bmatrix} \qquad (4\text{-}137)$$

Again, $\underset{\sim}{V}$ is (and in all cases considered will be) a vector of all ones,

$$\underset{\sim}{V} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} . \qquad (4\text{-}138)$$

It is desired, as before, to find the optimal control law, u*, such that the quadratic performance index (4-31) is minimized, and that the optimal system realizes a set of prescribed eigenvalues. The eigenvalues are given as:

$$s_1 = -2, \; s_2 = -5$$

$$s_3, s_4 = -6 \pm j3 \quad . \tag{4-139}$$

With the zeroes of the transfer function (4-136) the same, the transfer function of the desired system with the prescribed eigenvalues becomes:

$$T_4^*(s) = \frac{s^2+9s+34}{s^4+19s^3+138s^2+435s+450} \quad . \tag{4-140}$$

By continued fraction expansion of T*(s) the quotients obtained are:

$$\begin{array}{ll}
H_1 = 13.2353 & H_5 = -1077.43 \\
H_2 = .107635 & H_6 = .004834 \\
H_3 = -71.3206 & H_7 = 396.926 \\
H_4 = -.088175 & H_8 = -.024294 \quad ,
\end{array}$$

from which $\underset{\sim}{H}^*$ is:

$$H^* = -\begin{bmatrix} 1.42458 & -1.16702 & .06399 & -.32154 \\ 1.42458 & 5.12168 & -.28082 & 1.41114 \\ 1.42458 & 5.12168 & -5.48975 & 27.58655 \\ 1.4258 & 5.12168 & -5.48975 & 17.94349 \end{bmatrix} \tag{4-141}$$

Formulation of equation (4-50) yields in matrix notation:

$$\det = \begin{bmatrix} s\underset{\sim}{I}-\underset{\sim}{H} & \underset{\sim}{1} \\ \underset{\sim}{Q} & s\underset{\sim}{I}+\underset{\sim}{H}^T \end{bmatrix}^{+} = \det \begin{bmatrix} s\underset{\sim}{I}-H^{*} & \underset{\sim}{0} \\ \underset{\sim}{0} & sI+(H^{*})^T \end{bmatrix} \qquad (4\text{-}142)$$

which is to be solved for $\underset{\sim}{\tilde{Q}}$.

As before, we formulate the $\underset{\sim}{T}$, $\underset{\sim}{G}$ and $\underset{\sim}{W}$ matrices according to equations (4-53) and (4-54); (4-56), (4-57) and (4-58); and (4-60), respectively.

This yields:

$$\underset{\sim}{T} = \begin{bmatrix} 12.00 & 2.9064 & 2.7093 & 6.3500 \\ 0 & 9.6614 & 3.3893 & 6.2721 \\ 0 & 0 & 6.8313 & 6.8313 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (4\text{-}143)$$

$$\underset{\sim}{G} = \begin{bmatrix} 1 & 12.0000 & 46.5882 & 67.2941 \\ 1 & 9.6614 & 23.1534 & 0 \\ 1 & 6.8313 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad (4\text{-}144)$$

[+] the notation $\underset{\sim}{1}$ indicates an (nxn) matrix with all elements unity.

$$\underset{\sim}{W} = \begin{bmatrix} -1 & 12.0000 & -46.5882 & 67.2941 \\ -1 & 9.6614 & -23.1534 & 0 \\ -1 & 6.8313 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \qquad (4\text{-}145)$$

With the proper labels, (4-70) and (4-71),

for $\underset{\sim}{G}$:

$$\begin{array}{c|cccc|} & s^3 & s^2 & s^1 & s^0 \\ 1 & 1 & 12.0000 & 46.5882 & 67.2941 \\ 2 & 1 & 9.6613 & 23.1534 & 0 \\ 3 & 1 & 6.8313 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \end{array} \qquad (4\text{-}146)$$

for $\underset{\sim}{W}$:

$$\begin{array}{c|cccc|} & s^3 & s^2 & s^1 & s^0 \\ Q_{i1} & -1 & 12.0000 & 46.5882 & 67.2941 \\ Q_{i2} & -1 & 9.6613 & 23.1534 & 0 \\ Q_{i3} & -1 & 6.8313 & 0 & 0 \\ Q_{i4} & -1 & 0 & 0 & 0 \end{array} \qquad (4\text{-}147)$$

$\underset{\sim}{G}$ and $\underset{\sim}{W}$ contain all of the coefficients of $\tilde{Q}_{ij}s^{2k}$ resulting from the expansion of (4-142). Still to be found are the

coefficients of s^{2k} not involving any $\tilde{Q}_{ij}$, the σ_i's and σ_i*'s. Therefore, from (4-79) and (4-80):

$$(s^4+12s^3+48s^2+80s+48)(s^4-12s^3+48s^2-80s+48)$$

$$= s^8 - 48s^6 - 480s^4 - 1792s^2 + 2304, \qquad (4\text{-}148)$$

$$\sigma_0 = 2304 \qquad \sigma_3 = -48$$

$$\sigma_1 = -1792 \qquad \sigma_4 = 1 \qquad (4\text{-}149)$$

$$\sigma_2 = 480$$

and

$$(s^4+19s^3+138s^2+435s+450) \quad x$$

$$(s^4-19s^3+138s^2-435s+450)$$

$$= s^8-85s^6 + 3414s^4 - 65025s^2 + 202500 \quad , \qquad (4\text{-}150)$$

$$\sigma_0{}^* = 202500 \qquad \sigma_3{}^* = -85$$

$$\sigma_1{}^* = -65025 \qquad \sigma_4{}^* = 1 \quad . \qquad (4\text{-}151)$$

$$\sigma_2{}^* = 3414$$

Determination of the $\tilde{Q}_{ij}$'s results from equating coefficients of like powers of s from the expansion of the determinants in equation (4-142). The $\tilde{Q}_{ij}$'s are obtained in a successive method (4-65) by starting with the s^0 coefficients, then continuing by equating coefficients of s^{2k}, $k \underline{\varepsilon} [1, 2, \ldots n-1]$, in scandent fashion.

For s^0:

from $\underset{\sim}{G}$, the only non-zero element is g_{14}, corresponding to row 1, therefore, level 1 in $\underset{\sim}{W}$. The only non-zero element in level 1 of $\underset{\sim}{W}$ in the s^0 column is W_{141}. Equating coefficients yields:

$$(g_{14} \times W_{141})\, \tilde{Q}_{11} + \sigma_0 = \sigma_0^* \quad . \tag{4-152}$$

Solving:

$$\tilde{Q}_{11} = \frac{202500-2304}{(67.2941)(67.2941)} \cong 44.20803 \tag{4-153}$$

and from (4-67) and (4-68),

$$\begin{aligned} \tilde{Q}_{12} = \tilde{Q}_{21} &= -88.95327 \\ \tilde{Q}_{13} = \tilde{Q}_{31} &= 48.14819 \\ \tilde{Q}_{14} = \tilde{Q}_{41} &= -3.40294 \quad . \end{aligned} \tag{4-154}$$

for coefficients of s^2:

$$\sum_{j=1}^{1} g_{j4} \sum_{i=1}^{3} W_{i2j} \tilde{Q}_{ji} + \sum_{j=1}^{2} g_{j3} \sum_{i=1}^{2} W_{i3j} \tilde{Q}_{ji}$$

$$+ \sum_{j=1}^{3} g_{j2} \sum_{L=1}^{1} W_{i4j} \tilde{Q}_{ji} + \sigma_1 = +\sigma_1{}^* \quad . \qquad (4\text{-}155)$$

Solving:

$$\tilde{Q}_{22} \cong 296.94152 \qquad (4\text{-}156)$$

and from (4-67) and (4-68),

$$\tilde{Q}_{23} = \tilde{Q}_{32} = -263.70162$$

$$\tilde{Q}_{24} = \tilde{Q}_{42} = 55.71337 \qquad (4\text{-}157)$$

for coefficients of s^4:

$$\sum_{j=1}^{2} g_{j3} \sum_{i=1}^{4} W_{i1j} \tilde{Q}_{ji} + \sum_{j=1}^{3} g_{j2} \sum_{i=1}^{3} W_{12j} \tilde{Q}_{ji}$$

$$+ \sum_{j=1}^{4} g_{ji} \sum_{L=1}^{2} W_{i3j} \tilde{Q}_{ji} + \sigma_2 = \sigma_2{}^* \qquad (4\text{-}158)$$

Solving:

$$\tilde{Q}_{33} = 351.24159 \qquad (4\text{-}159)$$

and from (4-68),

$$\tilde{Q}_{34} = \tilde{Q}_{43} = -135.68816 \qquad (4\text{-}160)$$

for coefficients of s^6:

$$\sum_{j=1}^{4} g_{ji} \sum_{i=1}^{4} W_{ilj}\, \tilde{Q}_{ji} + \sigma_3 = \sigma_3{}^* \qquad (4\text{-}161)$$

$$\tilde{Q}_{44} = - \sum_{\substack{j=1 \\ i+j \neq 8}}^{4} \sum_{i=1}^{4} Q_{ij} + \sigma_3{}^* - \sigma_3 \qquad (4\text{-}162)$$

$$-\tilde{Q}_{44} = - \sum_{j=1}^{3} Q_{4j} + \sigma_3{}^* - \sigma_3 \qquad (4\text{-}163)$$

$$\tilde{Q}_{44} = 120.37773 \qquad (4\text{-}164)$$

$\tilde{\underset{\sim}{Q}}$ is now entirely known.

$$\underset{\sim}{\tilde{Q}} = \begin{bmatrix} 44.20803 & -88.95327 & 48.14819 & -3.40294 \\ -88.95327 & 296.94152 & -263.70162 & 55.71337 \\ 48.14819 & -263.70162 & 351.24159 & -135.68816 \\ -3.40294 & 55.71337 & -135.68816 & 120.37773 \end{bmatrix} \qquad (4\text{-}165)$$

From equation (4-91):

$$\underset{\sim}{Q} = \underset{\sim}{P}^T \underset{\sim}{\tilde{Q}} \underset{\sim}{P} = \begin{bmatrix} 200196 & 0 & 0 & 0 \\ 0 & 63233 & 0 & 0 \\ 0 & 0 & 2934 & 0 \\ 0 & 0 & 0 & 37 \end{bmatrix} \qquad (4\text{-}166)$$

Substituting matrices $\underset{\sim}{Q}$, R, $\underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{C}$ into the Riccati equation, and solving, yields the optimal feedback control law, u*;

$$u^* = -[402 \quad 355 \quad 90 \quad 7] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad (4\text{-}167)$$

where $G^* = -[402 \quad 355 \quad 90 \quad 7]$. (4-168)

The optimal closed loop system, $\dot{\underset{\sim}{x}} = (\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{G}^*)\underset{\sim}{x}$, in phase variable form is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -450 & -435 & -138 & -19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad (4\text{-}169)$$

which realizes the desired set of eigenvalues.

3. Higher Order Example (Seventh)

The previous examples represent an odd and an even ordered system, illustrating the minor differences in computational procedures. It is observed that for low order systems, the calculations can be done, relatively easily by hand. Higher order systems require, laborious and tedious computations. Appendix C provides a digital computer program which yields the weighting matrix, $\underset{\sim}{Q}$, with the only required input being the transfer functions of the known and desired eigenvalue systems.

The following seventh order example utilizes the results from the program given in Appendix C.

Consider the linear system given by its transfer function:

$$T_7(s) = \frac{249.435788}{s^7+9.0s^6+40.4s^5+116.8s^4+233.6s^3+323.2s^2 + 288.0s + 128.0} \quad . \qquad (4\text{-}170)$$

It is known that the characteristics of the desired system are such that its resulting transfer function is given by:

$$T_7^*(s) = \frac{249.435788}{s^7+15.4s^6+101.64s^5+372.68s^4+819.896s^3+1082.26272s^2+793.659328s+249.4357888} \tag{4-171}$$

The diagonal $\underset{\sim}{Q}$ matrix was determined from the computer program yielding:

$$\begin{aligned}
Q_{11} &= 45834.21273428\\
Q_{22} &= 89780.21841734\\
Q_{33} &= 55970.39786199\\
Q_{44} &= 19170.7157376\\
Q_{55} &= 3959.33664\\
Q_{66} &= 494.9776\\
Q_{77} &= 33.68 \quad .
\end{aligned} \tag{4-172}$$

$\underset{\sim}{Q}$, R and the state and output matrices representing the transfer function in equation (4-170) in phase variable form were substituted into the matrix Riccati equation. The optimal feedback control law, u*, resulting from solution of the Riccati equation is:

$$u^* = -121.435789\ x_1 - 505.659319\ x_2$$

$$-759.06269\ x_3 - 586.29596\ x_4 - 255.879974\ x_5$$

$$-61.2399914\ x_6 - 6.39999944\ x_7 \quad , \tag{4-173}$$

where the matrix G^* is:

$$G^* = -[121.435789\ \ 505.649319\ \ 759.06269\ \ 586.29596$$

$$255.879974\ \ 61.2399914\ \ 6.39999944]. \tag{4-174}$$

The optimal closed loop system,

$$\dot{\underset{\sim}{x}} = (\underset{\sim}{A} + \underset{\sim}{B}\underset{\sim}{G}^*)\ \underset{\sim}{x}$$

$$y = \underset{\sim}{C}\underset{\sim}{x} \quad ,$$

expressed as a transfer function is:

$$\frac{Y(s)}{u^*(s)} = \frac{249.435788}{s^7+15.39999974s^6+101.6399914s^5 + 372.679974s^4+819.89596s^3+ 1082.26289s^2+793.659319s+249.435789},$$

which realizes the desired transfer function with the prescribed eigenvalues.

Figures 4.2 through 4.13 show the impulse and step responses of the three previous examples, both before and after compensation.

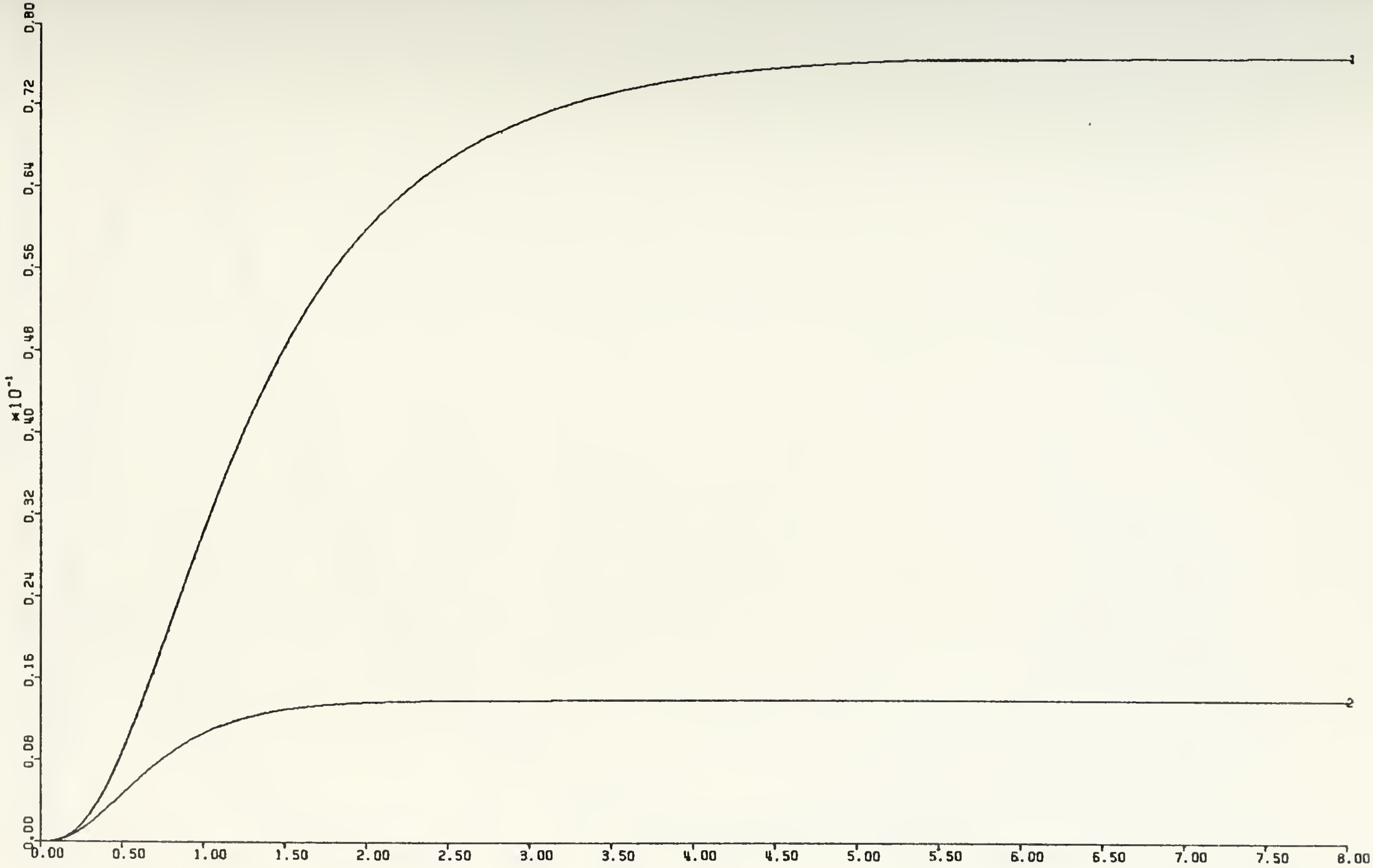

Figure 4.2. Unit Impulse Response of the Uncompensated (1) and Prescribed Eigenvalue (2) Third Order Systems

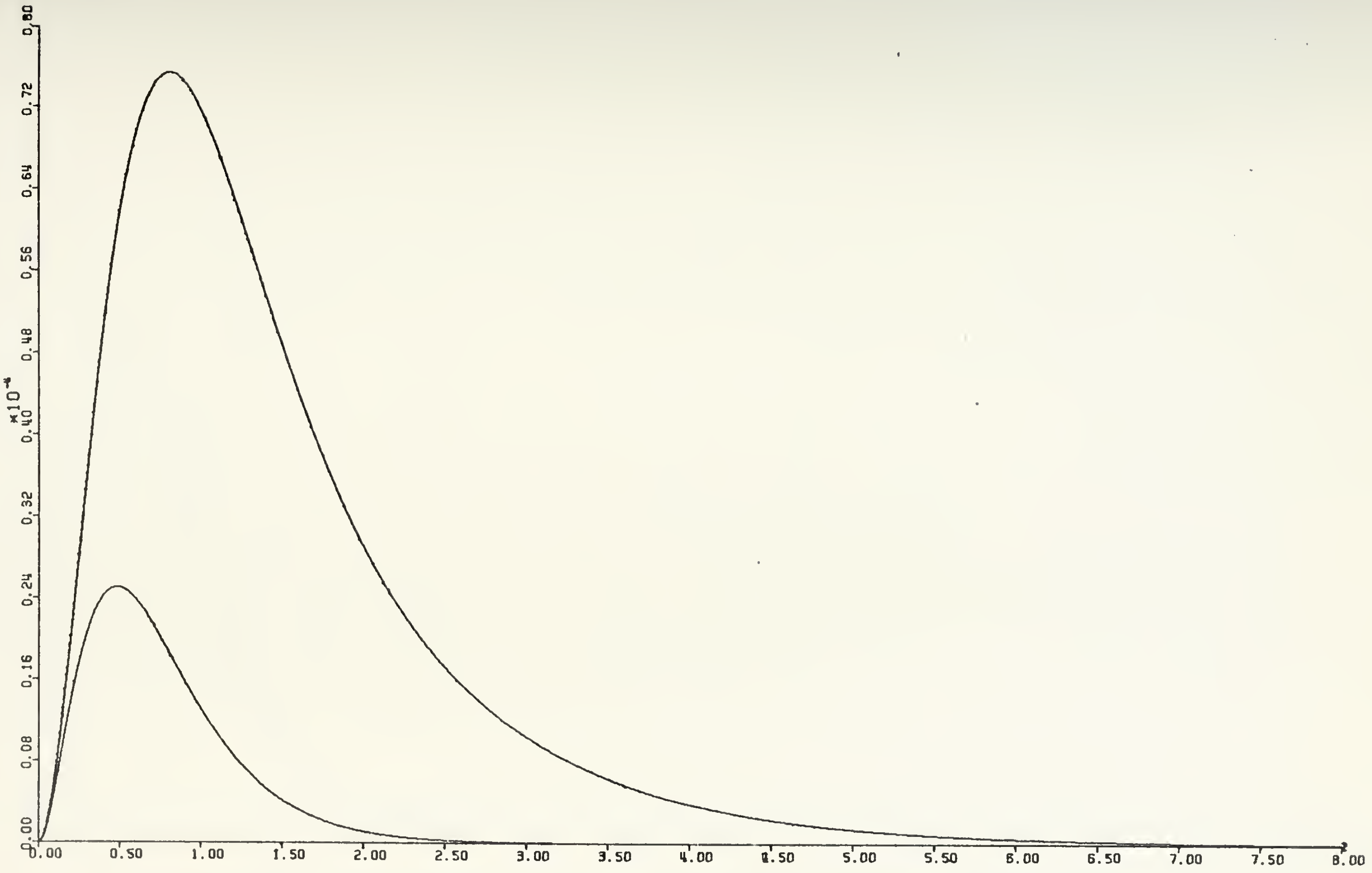

Figure 4.3. Unit Step Response of the Uncompensated (1) and Prescribed Eigenvalue (2) Third Order Systems

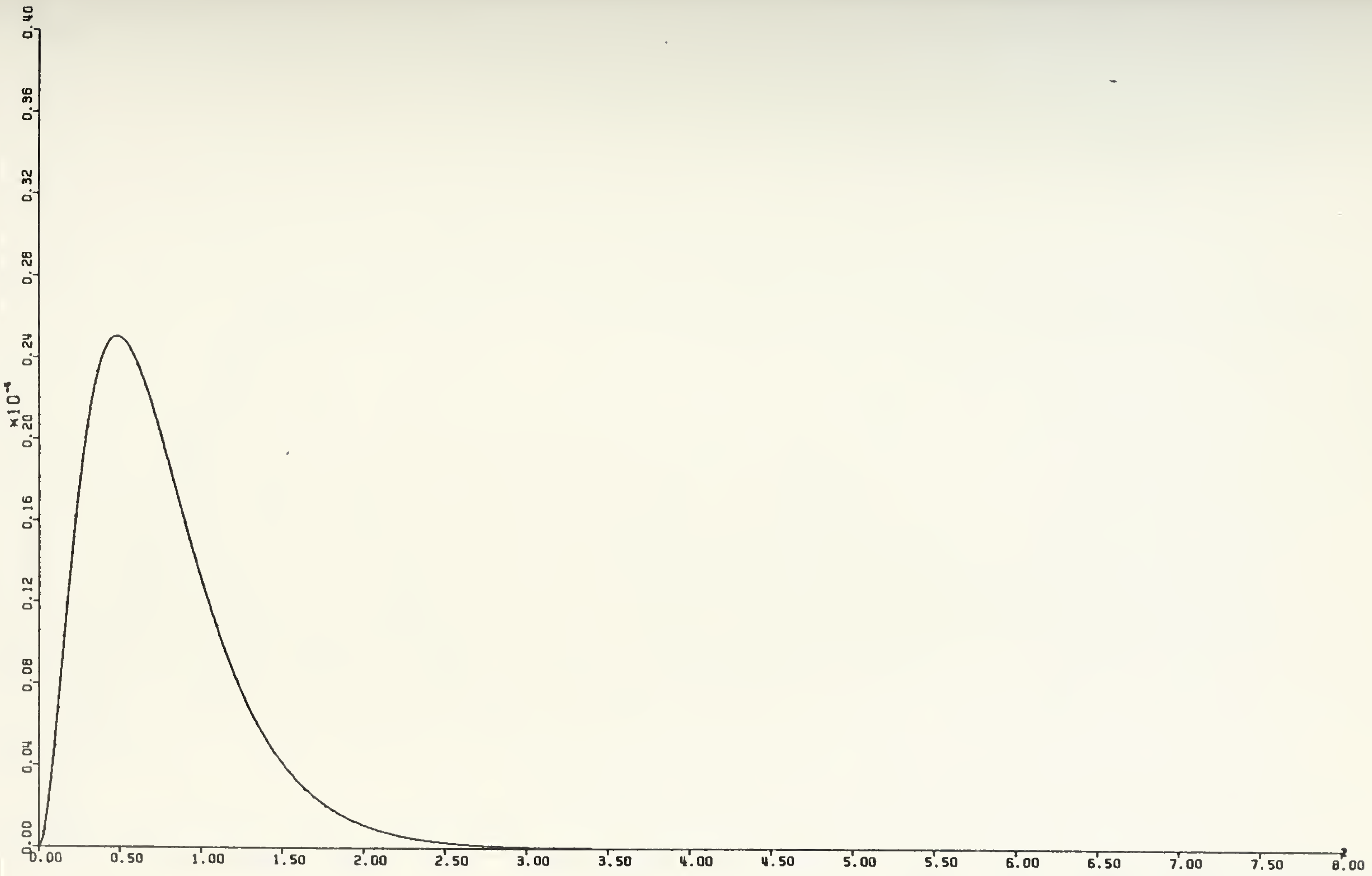

Figure 4.4. Unit Impulse Response of the Compensated (1) and Prescribed Eigenvalue (2) Third Order Systems

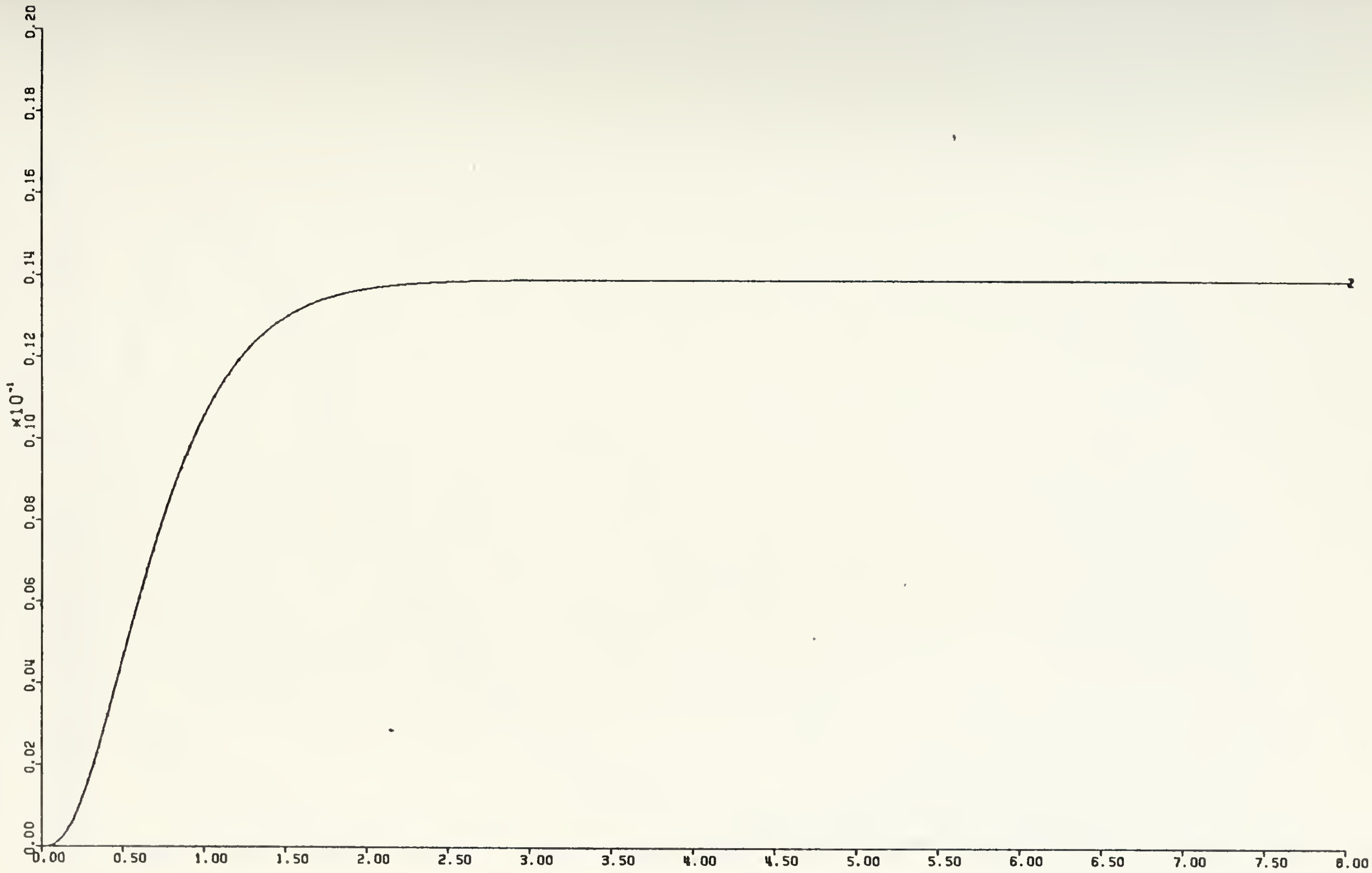

Figure 4.5. Unit Step Response of the Compensated (1) and Prescribed Eigenvalue (2) Third Order Systems

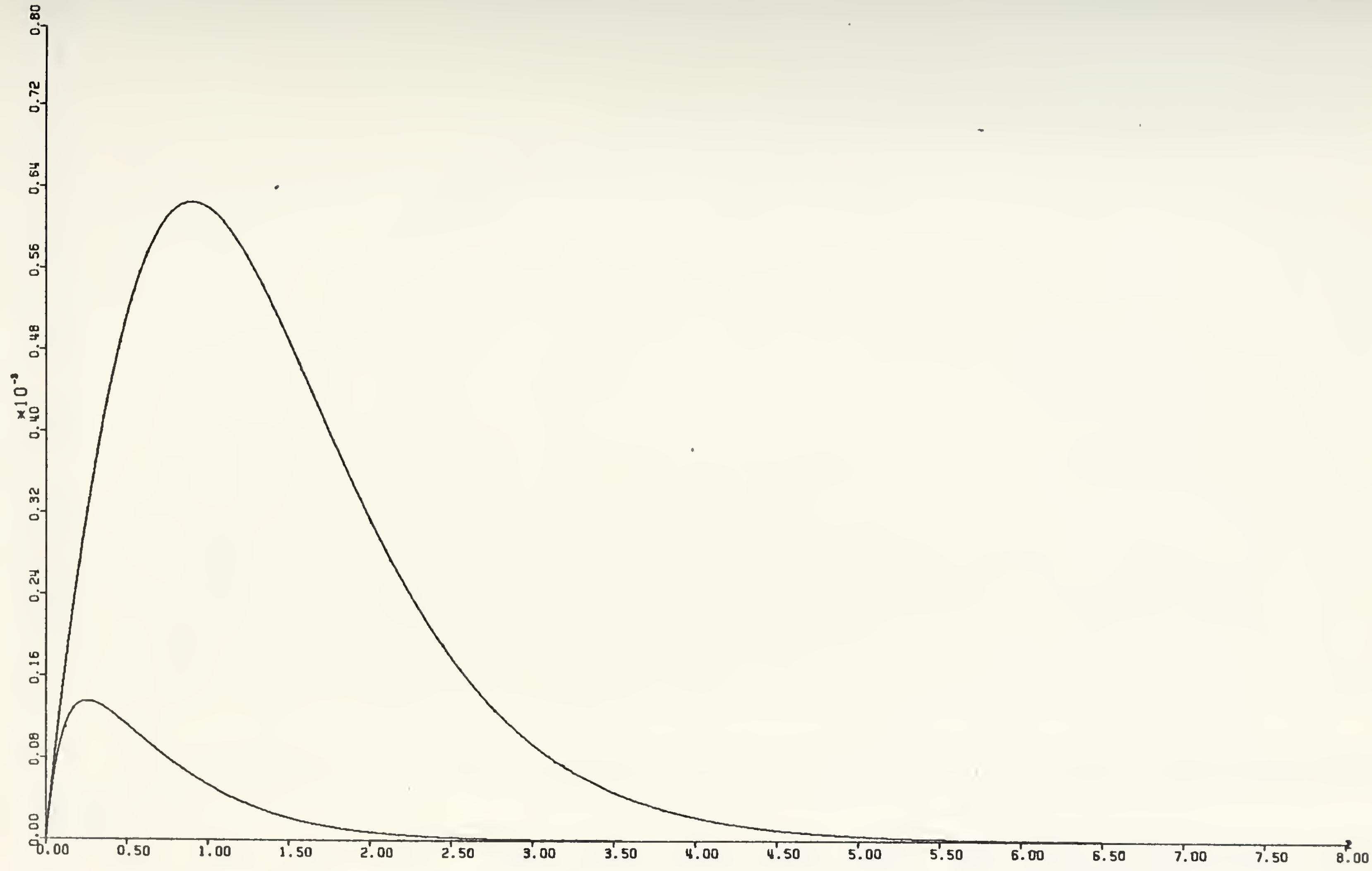

Figure 4.6. Unit Impulse Response of the Uncompensated (1) and Prescribed Eigenvalue (2) Fourth Order Systems

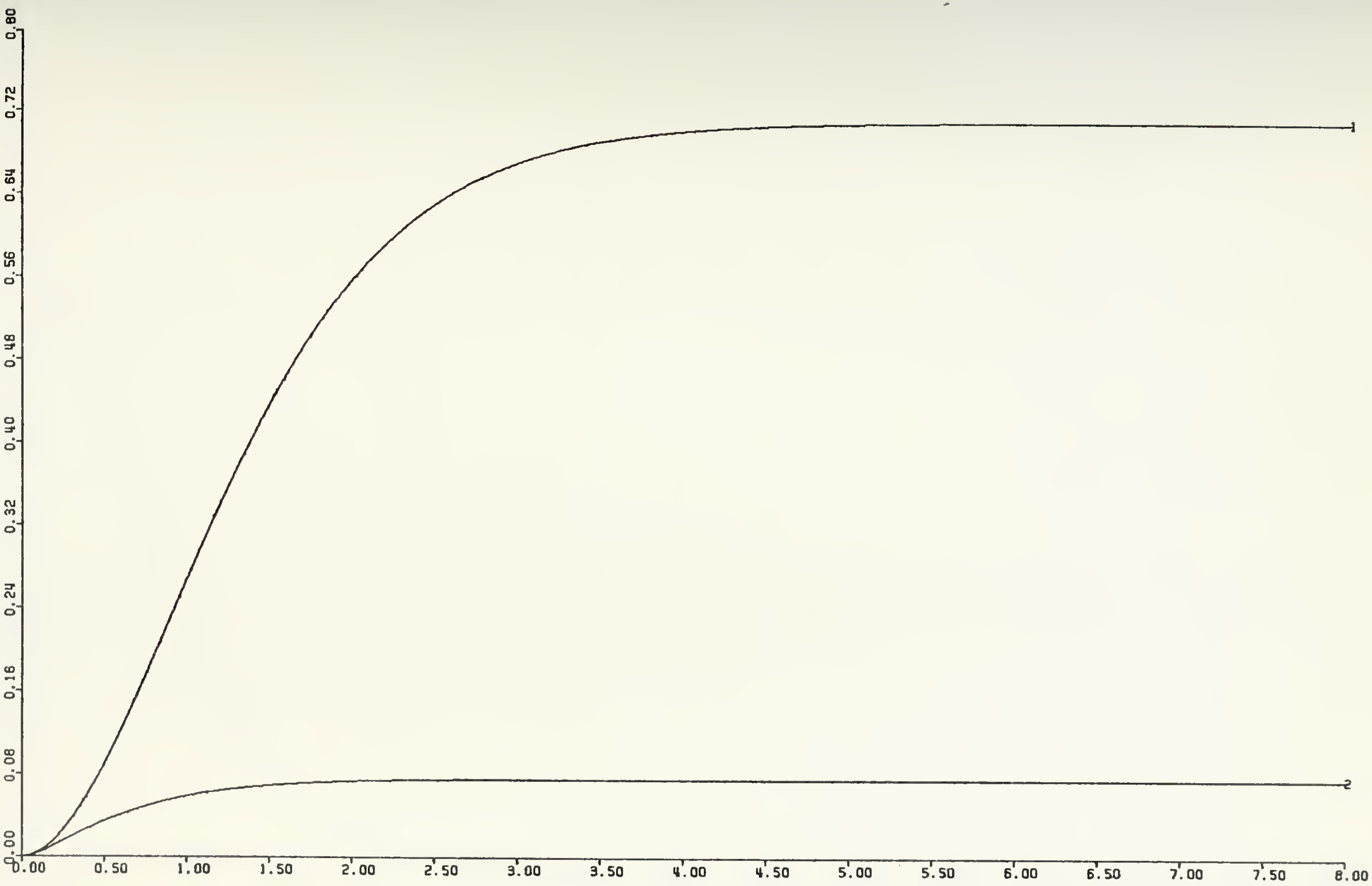

Figure 4.7. Unit Step Response of the Uncompensated (1) and Prescribed Eigenvalue (2) Fourth Order Systems

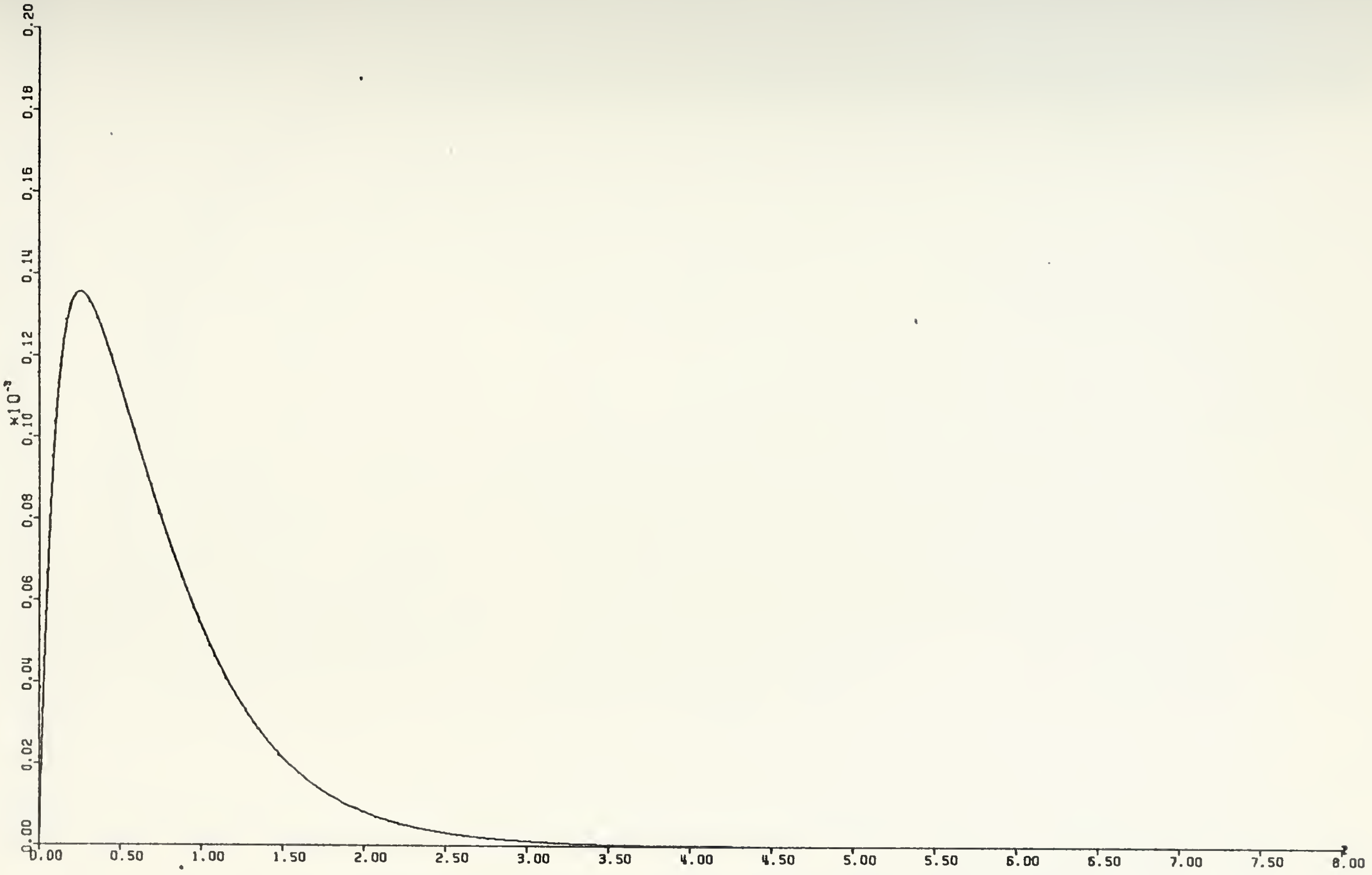

Figure 4.8. Unit Impulse Response of the Compensated (1) and Prescribed Eigenvalue (2) Fourth Order Systems

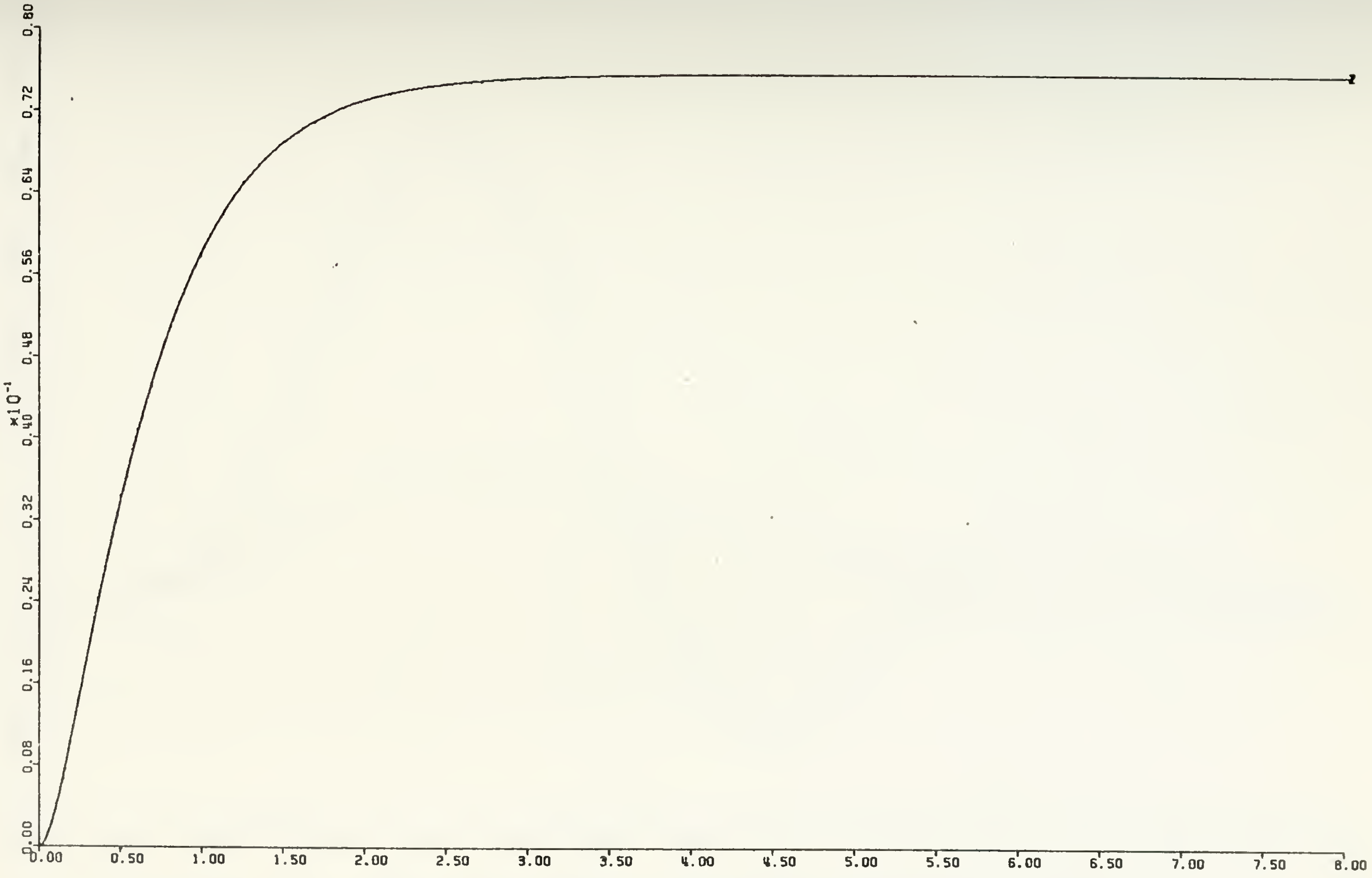

Figure 4.9. Unit Step Response of the Compensated (1) and Prescribed Eigenvalue (2) Fourth Order Systems

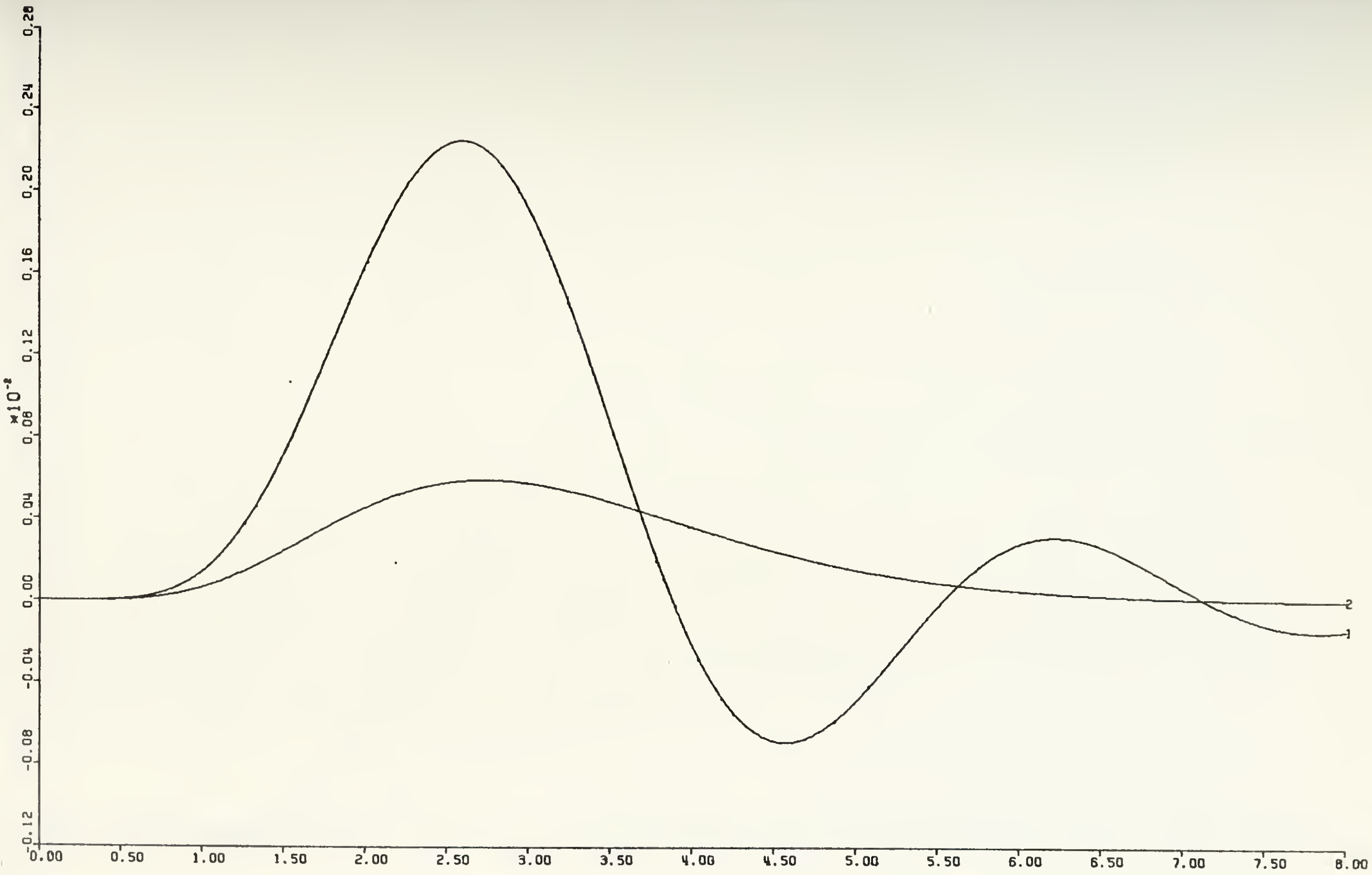

Figure 4.10. Unit Impulse Response of the Uncompensated (1) and Prescribed Eigenvalue (2) Seventh Order Systems

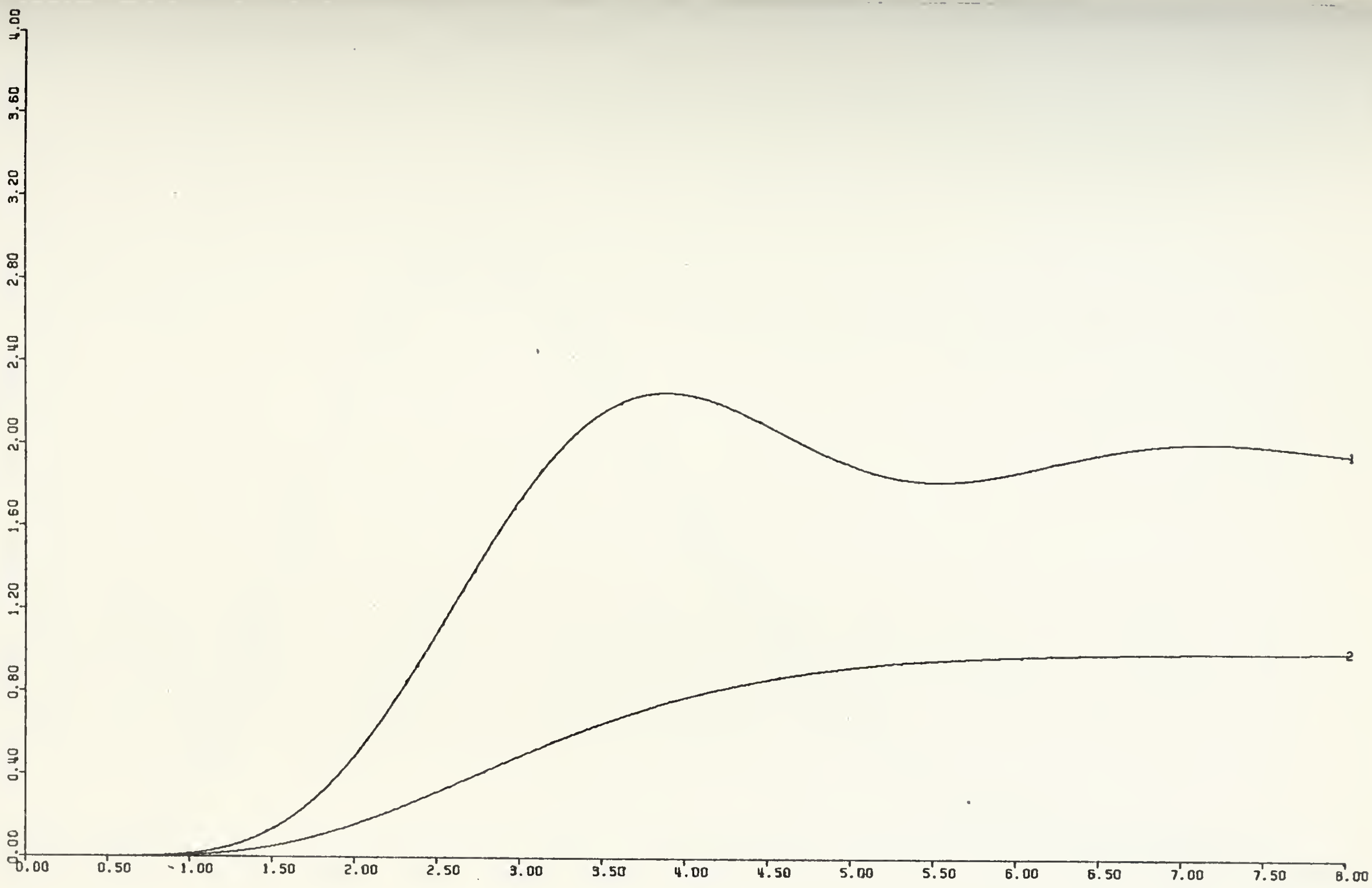

Figure 4.11. Unit Step Response of the Uncompensated (1) and Prescribed Eigenvalue (2) Seventh Order Systems

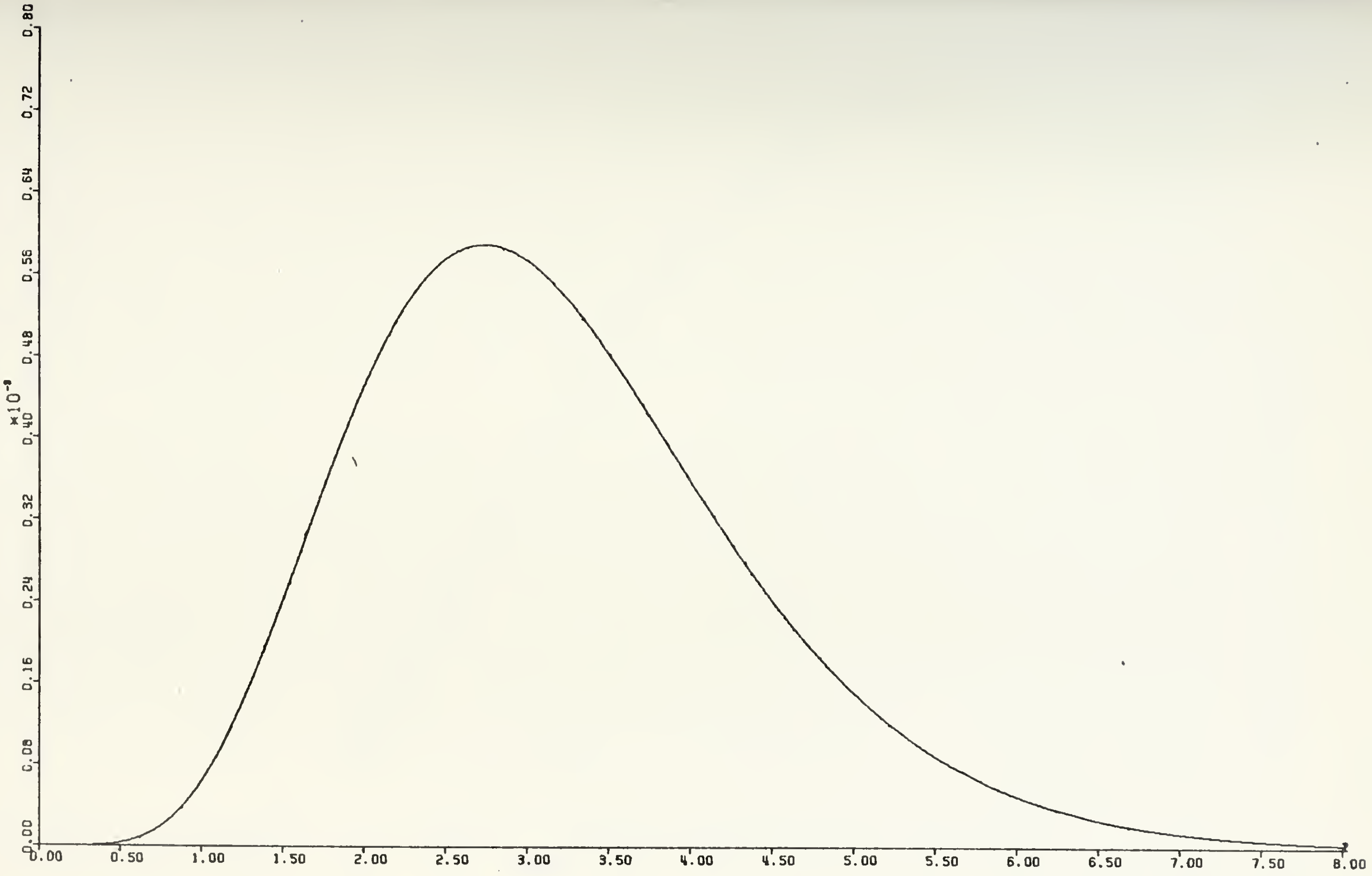

Figure 4.12. Unit Impulse Response of the Compensated (1) and Prescribed Eigenvalue (2) Seventh Order Systems

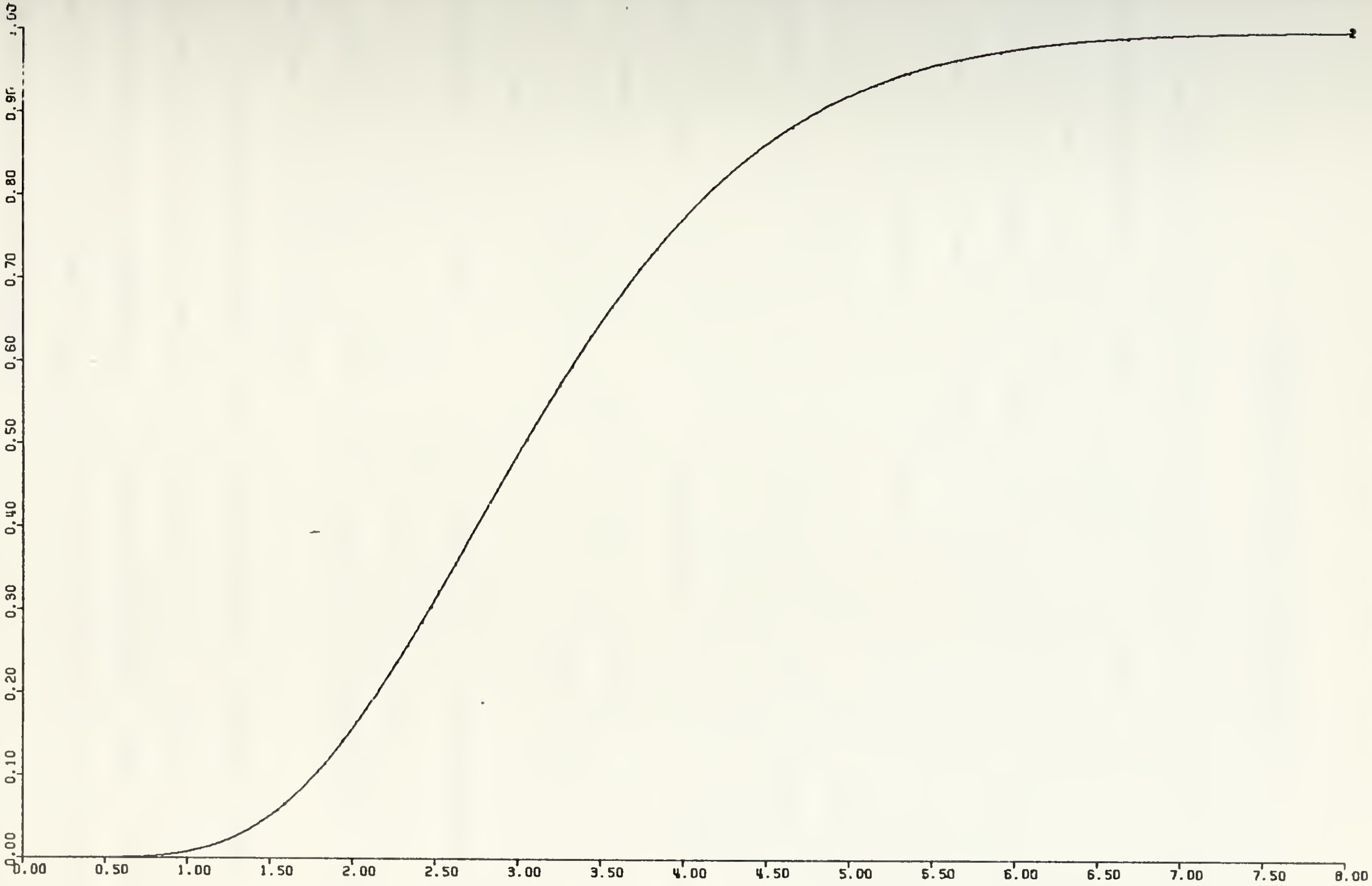

Figure 4.13. Unit Step Response of the Compensated (1) and Prescribed Eigenvalue (2) Seventh Order Systems

V. CONCLUSIONS AND DISCUSSION

A method has been shown for determination of the state weighting matrix in order to satisfy a prescribed set of eigenvalues through phase variable state feedback. From a strictly mathematical viewpoint, this technique requires only a knowledge of matrix algebra. Every attempt has been made to avoid the necessity of inverting a matrix. The introduction of Chapter IV made known the fact that previous developments in this area have suffered the main drawback of restriction. The author believes the method presented here, using Cauer Second Form, overcomes many of these restrictions. It presents a rational computational procedure for determination of the weighting matrix, $\underset{\sim}{Q}$; the system eigenvalues are only required to be in the left half of the complex plane as opposed to the left of a line parallel to the imaginary axis; and the method is no more complicated for multiple or complex eigenvalues than a system with linearly independent eigenvalues (or eigenvectors).

It should be noted that some authors "define" the eigenvalue(s) of a matrix to be only the real root(s) of the characteristic equation. In the development that has preceded in this thesis, all roots are considered eigenvalues of the associated matrix.

The algorithm derived in Chapter IV, is designed as a basis for future research. In particular, if the designer is working with nth order systems where n is relatively large, it may be advantageous to look at using mth order simplified models ($m<n$) of each system by a partitioning scheme similar to that in Chapter III. Again, it is emphasized that reduced order models do not necessarily yield stable systems. If the simplified system retains the basic characteristics of the original system, especially in steady state, then this would appear to be a reasonable approach.

A parallel approach could also be inventigated regarding multi-input multi-output (MIMO) systems, treating each element of the system transfer matrix as an individual transfer function.

To the author's knowledge, no work has been done in the digital or sampled data areas involving continued fraction theory. This area should be considered due to the increasing use and need for digital control systems.

These topics represent just a few of the areas available for future work.

APPENDIX A

Consider the system

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x} + \underset{\sim}{B}u, \qquad \text{(A-1)}$$

where $\underset{\sim}{x}$ is an n-dimensional state vector, u is the input function, and $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are time-invariant (nxn) and (nx1) matrices, respectively. The phase variable (canonical) system representation is defined as

$$\dot{\underset{\sim}{v}} = \hat{\underset{\sim}{A}}\underset{\sim}{v} + \hat{\underset{\sim}{B}}u, \qquad \text{(A-2)}$$

where $\underset{\sim}{v}$ is an n-dimensional state vector and

$$\hat{\underset{\sim}{A}} = \begin{bmatrix} 0 & 1 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 1 & \cdots\cdots & 0 \\ 0 & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots\cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots\cdots & -a_n \end{bmatrix}, \quad \hat{\underset{\sim}{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} . \qquad \text{(A-3)}$$

The systems represented in (A-1) and (A-2) are said to be equivalent if and only if there exists a non-singular matrix, $\underset{\sim}{K}$, such that

$$\underset{\sim}{x} = \underset{\sim}{K}\underset{\sim}{v} \qquad \text{(A-4)}$$

Kalman [23] has shown that a necessary and sufficient condition for such an equivalence to exist is that the system in (A-1) be completely controllable.

The controllability matrix of system (A-1) is defined by

$$\underset{\sim}{E} = [\underset{\sim}{B} \vdots \underset{\sim}{A}\underset{\sim}{B} \vdots \underset{\sim}{A}^2\underset{\sim}{B} \vdots \;\ldots\; \vdots \underset{\sim}{A}^{n-1}\underset{\sim}{B}] \qquad \text{(A-5)}$$

or in an equivalent manner

$$\underset{\sim}{E} = [e_1 \quad e_2 \quad \ldots \quad e_n], \qquad \text{(A-6)}$$

where the (nxl) vector e_i is recursively defined as

$$\underset{\sim}{e}_{i+1} = \underset{\sim}{A}\,\underset{\sim}{e}_i \;, \quad \underset{\sim}{e}_1 = \underset{\sim}{B} \;. \qquad \text{(A-7)}$$

The controllability matrix of system (A-2), $\hat{\underset{\sim}{E}}$, is defined in a similar manner with $\hat{\underset{\sim}{A}}$ and $\hat{\underset{\sim}{B}}$. Since there is only one control input, a necessary and sufficient condition for controllability is that the (nxn) matrix $\underset{\sim}{E}$ (or $\hat{\underset{\sim}{E}}$) have an inverse.

Silverman [20] has shown that if the system in (A-1) is controllable, then the transformation matrix, $\underset{\sim}{K}$, is determined by

$$\underset{\sim}{K} = \underset{\sim}{E}\,\hat{\underset{\sim}{E}}^{-1} \;, \qquad \text{(A-8)}$$

where

$$\underset{\sim}{\hat{E}}^{-1} = \begin{bmatrix} a_2 & a_3 & a_4 & \cdots\cdots & a_n & 1 \\ a_3 & a_4 & a_5 & \cdots\cdots & 1 & 0 \\ a_4 & a_5 & a_6 & \cdots\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_n & 1 & 0 & \cdots\cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots\cdots & 0 & 0 \end{bmatrix} , \qquad \text{(A-9)}$$

and $$\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = -\underset{\sim}{E}^{-1} \underset{\sim}{A}^{n} \underset{\sim}{B}. \qquad \text{(A-10)}$$

The elements of $\underset{\sim}{a}$ are the coefficients of the characteristic polynomial:

$$\det[S\underset{\sim}{I}-\underset{\sim}{A}] = \det[S\underset{\sim}{I}-\underset{\sim}{\hat{A}}] = S^{n} + \sum_{i=1}^{n} a_i S^{i-1} \qquad \text{(A-11)}$$

The matrix inversion in equation (A-10) can be avoided by using the Leverrier-Fadeev method for calculating the coefficients of the characteristic polynomial. Once the coefficients are known, $\underset{\sim}{\hat{E}}^{-1}$ is written by inspection.

Rane [24] presented a simplified procedure for finding the transformation matrix, $\underset{\sim}{K}$, requiring no matrix inversions. Substituting equation (A-4) into (A-1) and premultiplying equation (A-2) results in:

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{K}\underset{\sim}{v} + \underset{\sim}{B}u \qquad \text{(A-12)}$$

$$= \underset{\sim}{K}\underset{\sim}{\hat{A}}\underset{\sim}{v} + \underset{\sim}{K}\underset{\sim}{\hat{B}}u \quad . \qquad \text{(A-13)}$$

Comparison of equations (A-12) and (A-13) yields

$$\underset{\sim}{A}\underset{\sim}{K} = \underset{\sim}{K}\underset{\sim}{\hat{A}} \qquad \text{(A-14)}$$

and

$$\underset{\sim}{B} = \underset{\sim}{K}\underset{\sim}{\hat{B}} \quad . \qquad \text{(A-15)}$$

Partition $\underset{\sim}{K}$ into n column vectors, each (nxl), so that

$$\underset{\sim}{K} = [\underset{\sim}{k}_1 \;\; \underset{\sim}{k}_2 \;\; \ldots . \;\; \underset{\sim}{k}_n]. \qquad \text{(A-16)}$$

Substitution of (A-3) and (A-16) into (A-14) and (A-15) gives

$$\underset{\sim}{A}[\underset{\sim}{k}_1 \;\; \underset{\sim}{k}_2 \;\; \underset{\sim}{k}_3 \;\; \ldots . \;\; \underset{\sim}{k}_n] =$$

$$[\underset{\sim}{k}_1 \; \underset{\sim}{k}_2 \; \underset{\sim}{k}_3 \ldots .. \; \underset{\sim}{k}_n] \begin{bmatrix} 0 & 1 \;\ldots . & 0 & 0 \\ 0 & 0 \;\ldots . & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 \;\ldots .. & 0 & 1 \\ -a_1 & -a_2 \ldots . & a_{n-1} & -a_n \end{bmatrix} \qquad \text{(A-17)}$$

and

$$\underset{\sim}{B} = [\underset{\sim}{k}_1\ \underset{\sim}{k}_2\ \underset{\sim}{k}_3 \ldots\ \underset{\sim}{k}_n] \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} = \underset{\sim}{k}_n \, . \tag{A-18}$$

From (A-17) and (A-18)

$$\begin{aligned}
\underset{\sim}{k}_n &= B \\
\underset{\sim}{k}_{n-1} &= \underset{\sim}{A}\, \underset{\sim}{k}_n + \underset{\sim}{k}_n\, a_n \\
\underset{\sim}{k}_{n-2} &= \underset{\sim}{A}\, \underset{\sim}{k}_{n-1} + \underset{\sim}{k}_n\, a_{n-1} \\
&\ \vdots \\
\underset{\sim}{k}_2 &= \underset{\sim}{A}\, \underset{\sim}{k}_3 + \underset{\sim}{k}_n\, a_3 \\
\underset{\sim}{k}_1 &= \underset{\sim}{A}\, \underset{\sim}{k}_2 + \underset{\sim}{k}_n\, a_2 \, ,
\end{aligned} \tag{A-19}$$

or, in general,

$$\underset{\sim}{k}_{n-i} = \underset{\sim}{A}\, \underset{\sim}{k}_{n-i+1} + \underset{\sim}{k}_n\, a_{n-i+1} \tag{A-20}$$

for $i \varepsilon [1, 2, \ldots, n-1]$. The column vectors $\underset{\sim}{k}_1, \ldots, \underset{\sim}{k}_n$ are found in a simple recursive manner and completely determine the transformation matrix.

APPENDIX B

INVERSION OF CAUER I AND CAUER II FORMS

This program was written in FORTRAN IV, and requires minimal input. The only information required is:

1. the order of the system
2. which inversion is required
3. the quotients from the continued fraction expansion.

Multiple data sets are possible, and input in the following format:

Card	Columns	Description	Format
1	1-3	M = the desired order transfer function	I3
2	1-20	h_1, the first quotient of either Cauer I or Cauer II continued fraction expansion	D20.13
	21-40	h_2, the second quotient	D20.13
	41-60	h_3, the third quotient	D20.13
	61-80	h_4, the fourth quotient	D20.13
⋮	⋮	⋮	⋮
N	1-20	h_{4M-7}, the (4N-7)th quotient such that $4N-7 \leq 2M$	D20.13
	21-40	h_{4N-6}, $4N-6 \leq 2M$	D20.13
	41-60	h_{4N-5}, $4N-5 \leq 2M$	D20.13
	61-80	h_{4N-4}, $4N-4 \leq 2M$	D20.13
		Four quotients per card until 2*M quotients have been input, where M is the system order. Assume this is the Lth card. The (L+1)th data card begins the second data set.	

L+1	1-3	M=system order	I3
	4-6	K=1 for Cauer I and K=2 for Cauer II inversion	I3
.	.	.	.
.	.	.	.
.	.	.	.

The computer program has been written to handle up to 20th order systems ($M \leq 20$). If it is required to work with higher order systems, only one card change must be made. The specification statement is modified to read:

```
REAL*8 A(N), B(N), C(N)/N*0./, D(N)/N*0./, DZERO
```

where N is an integer no larger than 999. This restriction can be lifted by changing statement 2 to read:

```
2 FORMAT(2IR)
```

where R is the mantissa of $\log_{10}(N)$. The REAL*8 in the specification statement indicates that all following variables and arrays are real valued and in double precision. Modification to either single or extended precision would require changes in all format statements. If this is desireable, the user should consult references [25] and [26].

Execution time has shown to be less than .18 seconds for systems of order 10 or less.

```
// EXEC FORTCLG
//FORT.SYSIN DD *
      REAL*8 A(20),B(20),C(20)/20*0./,D(20)/20*0./,DZERO
C
C     ... READ IN SYSTEM ORDER, & WHICH CAUER INVERSION ...
C
   16 READ(5,1) M,K
    1 FORMAT(2I3)
C
C     ... READ IN QUOTIENTS FROM CONT. FRAC. EXPANSION ...
C
      READ(5,2) (A(I),B(I),I=1,M)
    2 FORMAT(4D20.13)
C
C     ... DETERMINE IF INVERSION IS CAUER I OR CAUER II ...
C
      IF(K.EQ.1) GO TO 10
C
C     ... CAUER II INVERSION ...
C     ... INITIALIZATION ...
C
      C(M) = B(M)
      M1 = M-1
      D(M1) = 1.0
      D(M) = A(M)*C(M)
C
C     ... ITERATION ...
C
      DO 3 I=1,M1
      L = I+1
      K = N-L
      DO 4 J=1,L
      KJ = K+J
      KL = KJ-1
      KP = KJ+1
      C(KJ) = B(K)*D(KJ)
      IF(J.NE.L) C(KJ) = C(KJ)+C(KP)
      IF(KL.EQ.0) GO TO 5
      D(KL) = A(K)*C(KL)+D(KJ)
      GO TO 4
    5 DZERO = 1.0
    4 CONTINUE
      D(M) = A(K+1)*C(M)
    3 CONTINUE
      GO TO 11
C
C     ... CAUER II INVERSION ...
C     ... INITIALIZATION ...
C
   10 C(M) = B(M)
      MM = M-1
      D(MM) = A(M)*C(M)
      D(M) = 1.0
C
C     ... ITERATION ...
C
      DO 6 I=1,MM
      IP = I+1
      MI = M-IP
      DO 7 J=1,IP
      MJ = MI+J
      ML = MJ-1
      MP = MJ+1
      C(MJ) = B(MI)*D(MJ)+C(MJ)
      IF(ML.EQ.0) GO TO 8
      D(ML) = A(MI)*C(MJ)+D(ML)
      GO TO 7
```

```
    8 DZERO = 1.0
    7 CONTINUE
    6 CONTINUE
C
C     ... WRITE OUT TRANSFER FUNCTION IN TERMS OF
C         NUMERATOR AND DENOMINATOR COEFFICIENTS
C         WITH APPROPRIATE POWER OF S ...
C
   11 WRITE(6,12)
   12 FORMAT(///11X,'NUMERATOR',15X,'DENOMINATOR',
     110X,'POWER OF S')
      WRITE(6,13) DZERO,M
   13 FORMAT(///31X,D20.13,10X,I2)
      DO 14 I=1,M
      MMI = M-I
      WRITE(6,15) C(I),D(I),MMI
   14 CONTINUE
   15 FORMAT(///6X,D20.13,4X,D20.13,10X,I2)
C
      GO TO 16
END
C
C     FOR ACTUAL RUN THIS CARD IS  /*  IN COLUMNS 1 AND 2
C
//GO.SYSIN DD *
C
C     DATA INPUT
C
```

APPENDIX C

DETERMINATION OF WEIGHTING MATRIX ($\underset{\sim}{Q}$) FOR PRESCRIBED EIGENVALUES

This FORTRAN IV program was used exclusively on the Naval Postgraduate School's IBM 360/67 digital computer and includes the associated job control language statements. The program consists of a main program and nine subroutine subprograms. The purpose of each subroutine is delineated below.

SUBROUTINE	DESCRIPTION
READ	read in coefficients of numerator and denominator polynomials of both transfer functions, and places each system in phase variable form.
RAMAT	determines the Routh array matrix and the transformation matrix, $\underset{\sim}{P}$.
MULTPH	multiples two matrices, $\underset{\sim}{Y}$ and $\underset{\sim}{Z}$, and gives the resulting matrix $\underset{\sim}{YZ}$.
HMATRX	determines the quotients of continued fraction expansion, H1(H2), and the state matrix in Cauer II form, $\underset{\sim}{HH1}(\underset{\sim}{HH2})$.
POLYNM	determines the product $\det(S\underset{\sim}{I}-\underset{\sim}{A})x\det(S\underset{\sim}{I}+\underset{\sim}{A}^T)$
HELP	computes the matrices $\underset{\sim}{G}$, $\underset{\sim}{T}$, and $\underset{\sim}{W}$ as given in Chapter IV.
QTILDA	computes the matrices, $\underset{\sim}{\tilde{Q}}$, and $\underset{\sim}{Q}$ from results of subroutine HELP.
QPIJ	determines the off diagonal elements, $\tilde{q}_{ij}$, of the matrix $\underset{\sim}{\tilde{Q}}$.
WRITE	writes all two-dimensional matrices.

The input required has been reduced to a minimum. Multiple data sets are possible; and input as indicated below:

Card	Columns	Description	Format
1	1-3	N; the system order	I3
2 . . .	1-16, 17-32, 33-48 ...	a_{n-1} for $i\varepsilon[1,2,\ldots,n]$; the denominator coefficients of the known transfer function, for an Nth order system, L cards are required, where L =K+1 and K is the integral part of N/5.	F16.8
L+2	1-16, 17-32, 33-48 ...	b_{n-1} for $i\varepsilon[1,2,\ldots,n]$; the numerator coefficients of the known transfer function, L cards required.	F16.8
2L+2	1-16, 17-32, 33-48, ...	α_{n-1} for $i\varepsilon[1,2,\ldots,n]$; the denominator coefficients of the transfer function with prescribed eigenvalues.	F16.8
3L+2	1-16, 17-32, 33-48, ...	β_{n-1} for $i\varepsilon[1,2,\ldots,n]$; the numerator coefficients of the transfer function with prescribed eigenvalues.	F16.8
		for multiple data sets, repeat the same prodecure. Each data set requires 4L+1 cards.	

This program has been written to accept systems up through 20th order. To increase the capability of the program, only the dimension statements and the second continuation card of the equivalence statement require modification. The system order capability can be increased to 50. Beyond 50th order, the program requires an excessive amount of storage space (>510K bytes). Even this limitation is easily overcome by removing the four cards between statements 1000 and 1001.

Any other modifications (i.e., single or extended precision) require extensive changes to all subprograms. In this case, the user should consult [25] and/or [26]. It is recommended that an object deck or disk storage be used when available as compilation time is approximately 70-80% of total CPU time.

```
// EXEC FORTCLG
//FORT.SYSIN DD *
      IMPLICIT REAL*8 (A-H,O-Z), INTEGER (I-N,$)
C
      DIMENSION A(20,20), AA(20,20), AID(20,20), A1(20,20), B(20),
     1BB(20), B1(20), C(20), CC(20), CD(20,20), CR(20), C1(20), DB(20),
     2DD(20), DE(20), DR(20), D1(20), DC1(21), DC2(21), E(21), E1(21),
     3E2(21), F(20,20,20), H(20,20), HMTX(20,20), H1(40), H2(40),
     4HH1(20,20), HH2(20,20), P(20,20), P1(20,20), P2(20,20), Q(20),
     5PT1(20,20), PT1Q(20,20), PT1QP1(20,20), QP(20,20), R(42,21),
     6RHO(20,20), RACOL1(42), RACOL2(42), RAM1(42,21), RAM2(42,21),
     7V(20,20), Y(20,20), YZ(20,20), Z(20,20), RA(42), RB(40), RC(42)
C
      COMMON HH1 /ONE/ E1, E2 /TWO/ QP, P1 /THREE/ V, F /FOUR/ HH2
C
      EQUIVALENCE (B(1),BB(1)), (RAM1(1),RAM2(1),PT1Q(1)),
     1(RACOL1(1),RACOL2(1),QP(1)), (A(1),AA(1),V(1)),
     2(RAM1(401),RAM2(401),PT1QP1(1)), (DD(1),DE(1))
C
C
C
 8888 READ(5,1000) N
 1000 FORMAT(I3)
      IF(N.LE.50) GO TO 1001
      WRITE(6,1002)
 1002 FORMAT('0 N IS GREATER THAN 50, PROGRAM WILL NOT RUN')
      STOP
C
 1001 CALL READ(N,A,B,C,DD)
      CALL RAMAT(N,RAM1,P1,C,DD,RACOL1)
C
      DO 71 I=1,N
      DO 72 K=1,N
      PT1(I,K) = P1(K,I)
   72 CONTINUE
   71 CONTINUE
C
      CALL HMATRX(N,RACOL1,H1,HH1)
      WRITE(6,4401)
 4401 FORMAT('0',10X,'THE ''H'' MATRIX IS'///)
      CALL WRITE(N,HH1)
      CALL POLYNM(N,DD,E1)
C
C
C
      CALL READ(N,AA,BB,CC,DE)
      CALL RAMAT(N,RAM2,P2,CC,DE,RACOL2)
      CALL HMATRX(N,RACOL2,H2,HH2)
```

```
      WRITE(6,4401)
      CALL WRITE(N,HH2)
      CALL POLYNM(N,DE,E2)
C
C
C
      CALL HELP(N)
C
C
C
      CALL QTILDA(N)
C
C
C
      CALL MULTPH(PT1,QP,PT1Q,N,N,N)
      CALL MULTPH(PT1Q,P1,PT1QP1,N,N,N)
C
C
C
      DO 998 I=1,N
      DO 999 K=1,N
      IF(DABS(PT1QP1(I,K)).LE.1.0D-06) GO TO 999
      IF(I.EQ.K) GO TO 999
      WRITE(6,991) I,K,PT1QP1(I,K)
  991 FORMAT('0  Q(',I2,',',I2,') =  ',D20.13)
  999 CONTINUE
  998 CONTINUE
C
C
C
      WRITE(6,995)
  995 FORMAT(///10X,'THE DIAGONAL ELEMENTS OF THE ''Q'' MATRIX ARE'//)
      WRITE(6,994) (I,I,PT1QP1(I,I),I=1,N)
  994 FORMAT(//20X,'Q(',I2,',',I2,') =  ',D20.13)
C
      GO TO 8888
      END
```

```
      SUBROUTINE READ(N,A1,B1,C1,D1)
      IMPLICIT REAL*8 (A-H,O-Z), INTEGER (I-N,$)
C
      DIMENSION A1(20,20), B1(20), C1(20), D1(20)
C
C
C
      NM1 = N-1
      DO 51 I=1,NM1
      DO 52 J=1,N
      A1(I,J) = 0.0D+00
      IF(J.EQ.I+1) A1(I,J) = 1.0D+00
   52 CONTINUE
   51 CONTINUE
C
      READ(5,1,ERR=60) (A1(N,N-K+1),K=1,N)
      DO 53 L=1,N
      D1(L) = A1(N,L)
      A1(N,L) = -A1(N,L)
   53 CONTINUE
C
      DO 54 M=1,NM1
      B1(M) = 0.0D+00
   54 CONTINUE
C
      B1(N) = 1.0D+00
      READ(5,1,ERR=60) (C1(N-I+1),I=1,N)
    1 FORMAT(5F16.8)
      GO TO 55
   60 WRITE(6,101)
  101 FORMAT('0 ERROR IN INPUT DATA')
      STOP
C
   55 CONTINUE
C
      WRITE(6,3000)
 3000 FORMAT('1',10X,'THE ''A'' MATRIX IS'///)
      CALL WRITE(N,A1)
      WRITE(6,3002)
 3002 FORMAT(//10X,'THE ''B'' MATRIX IS',25X,'THE ''C'' MATRIX IS'/)
      WRITE(6,3003) (B1(K),C1(K), K=1,N)
 3003 FORMAT(//15X,F5.2,32X,D20.13)
      WRITE(6,3004)
 3004 FORMAT(///35X,'********************',20X,'********************')
C
      RETURN
      END
```

```
      SUBROUTINE RAMAT(N,R,P,CR,DR,RC)
      IMPLICIT REAL*8 (A-H,O-Z), INTEGER (I-N,$)
C
      DIMENSION R(42,21), CR(20), DR(20), P(20,20), RC(42)
C
C
C
      NT1 = N+1
      N22 = 2*NT1
      DO 61 I=1,N
      R(1,I) = DR(I)
      R(2,I) = CR(I)
   61 CONTINUE
C
      R(1,NT1) = 1.0D+00
      R(2,NT1) = 0.0D+00
      DO 62 J=3,N22
      DO 63 K=1,NT1
      R(J,K) = 0.0D+00
   63 CONTINUE
   62 CONTINUE
C
      DO 64 L=3,N22
      K = (L-1)/2
      KK = 2*K
      DO 65 M=1,NT1
      IF((M.EQ.N-K+2).AND.((L.EQ.KK+1).OR.(L.EQ.KK+2))) GO TO 64
      R(L,M) = (R(L-1,1)*R(L-2,M+1)-R(L-2,1)*R(L-1,M+1))/R(L-1,1)
   65 CONTINUE
   64 CONTINUE
C
      DO 66 J=1,N
      J21 = 2*J + 1
      DO 67 K=1,N
      IF(J.GT.K) GO TO 68
      KJ1 = K-J+1
      P(J,K) = R(J21,KJ1)
      GO TO 67
   68 P(J,K) = 0.0D+00
   67 CONTINUE
   66 CONTINUE
C
      DO 69 M=1,N
      IF(DABS(P(M,N)-1.0D+00).LE.1.0D-10) GO TO 69
      WRITE(6,201) M, P(M,N)
  201 FORMAT('0 P(',I2,'N) =  ',D20.13,' NOT EQUAL ONE'//)
      STOP
C
```

```
   69 P(M,N) = 1.0D+00
C
      DO 73 I=1,N22
      RC(I) = R(I,1)
   73 CONTINUE
C
      WRITE(6,3007)
 3007 FORMAT(//'0',10X,'THE ''P'' MATRIX IS'///)
      CALL WRITE(N,P)
C
      RETURN
      END
```

```
      SUBROUTINE MULTPH(Y,Z,YZ,N1,N2,N3)
      IMPLICIT REAL*8 (A-H,O-Z), INTEGER (I-N,$)
C
      DIMENSION Y(20,20), Z(20,20), YZ(20,20), Q(20)
C
C
C
      DO 91 I=1,N1
      DO 92 J=1,N3
      DO 93 K=1,N2
      Q(K) = Y(I,K)*Z(K,J)
      L = K
      M = K-1
      IF(M.NE.0) GO TO 94
      Q(L) = Q(K)
      GO TO 93
   94 Q(L) = Q(K) + Q(M)
      IF(L.NE.N2) GO TO 93
      YZ(I,J) = Q(L)
   93 CONTINUE
   92 CONTINUE
   91 CONTINUE
C
      RETURN
      END
```

```
      SUBROUTINE HMATRX(N,RA,RB,HMTX)
      IMPLICIT REAL*8 (A-H,O-Z), INTEGER (I-N,$)
C
      DIMENSION RA(42), RB(42), HMTX(20,20)
C
C
C
      NN = 2*N
      DO 95 I=1,NN
      RB(I) = RA(I)/RA(I+1)
   95 CONTINUE
C
      DO 96 I=1,N
      I21 = 2*I - 1
      TEMP = 0.0
      DO 97 J=1,I21,2
      TEMP = TEMP + RB(J)
   97 CONTINUE
      DO 98 K=1,N
      K2 = 2*K
      IF(I.GT.K) GO TO 99
      HMTX(I,K) = -TEMP*RB(K2)
      GO TO 98
   99 HMTX(I,K) = HMTX(K,K)
   98 CONTINUE
   96 CONTINUE
C
      RETURN
      END
```

```
      SUBROUTINE POLYNM(N,DB,E)
      IMPLICIT REAL*8 (A-H,O-Z), INTEGER (I-N,$)
C
      DIMENSION DB(20), E(21), DC1(21), DC2(21)
C
C
C
      NT1 = N+1
      DO 710 I=1,N
      DC1(I) = DB(I)
  710 CONTINUE
C
      DC1(NT1) = 1.0D+00
      DO 711 J=1,NT1,2
      K = NT1-J+1
      DC2(K) = DC1(K)
      IF(K.EQ.1) GO TO 712
      L = K-1
      DC2(L) = -DC1(L)
  711 CONTINUE
C
  712 NT1T2 = 2*NT1
      DO 714 K=2,NT1T2,2
      IHALF = K/2
      E(IHALF) = 0.0D+00
      DO 715 L=1,NT1
      DO 716 M=1,NT1
      LPM = L+M
      IF(LPM.NE.K) GO TO 716
      E(IHALF) = E(IHALF) + (DC1(L)*DC2(M))
  716 CONTINUE
  715 CONTINUE
  714 CONTINUE
C
      RETURN
      END
```

```
      SUBROUTINE HELP(N)
      IMPLICIT REAL*8 (A-H,O-Z), INTEGER (I-N,$)
C
      DIMENSION H(20,20), AID(20,20), V(20,20), F(20,20,20)
C
      COMMON H/THREE/ V, F
C
C
C
      NM1 = N-1
      DO 451 I=1,NM1
      IP1 = I+1
      DO 452 J=IP1,N
      AID(I,J) = H(I,J) - H(J,J)
  452 CONTINUE
  451 CONTINUE
C
      DO 453 K=1,NM1
      ELSE = 0.0D+00
      KP1 = K+1
      DO 454 L=KP1,N
      ELSE = ELSE + AID(K,L)
  454 CONTINUE
      AID(K,K) = ELSE
  453 CONTINUE
C
      DO 464 I=2,N
      K = N-I+1
      DO 465 J=1,N
      IF(J.GT.K) V(I,J) = 0.0D+00
  465 CONTINUE
  464 CONTINUE
C
      DO 455 K=1,N
      V(K,1) = 1.0D+00
  455 CONTINUE
C
      DO 456 L=1,NM1
      V(L,2) = AID(L,L)
  456 CONTINUE
C
      DO 457 J=3,N
      NMJP1 = N-J+1
      DO 458 I=1,NMJP1
      ETA = 0.0D+00
      NMJP2 = N-J+2
      IP1A = I+1
      DO 459 K=IP1A,NMJP2
```

```
      JM1 = J-1
      ETA = ETA + (AID(I,K)*V(K,JM1))
  459 CONTINUE
      V(I,J) = ETA
  458 CONTINUE
  457 CONTINUE
C
      DO 460 K=1,N
      DO 461 J=1,N
      NMJP1A = N-J+1
      DO 462 I=1,NMJP1A
      IF(MOD(J,2).NE.0) GO TO 463
      F(I,J,K) = V(I,J)
      GO TO 462
  463 F(I,J,K) = -V(I,J)
  462 CONTINUE
  461 CONTINUE
  460 CONTINUE
C
      DO 466 L=1,N
      DO 467 I=2,N
      K = N-I+1
      DO 468 J=1,N
      IF(J.GT.K) F(I,J,L) = 0.0D+00
  468 CONTINUE
  467 CONTINUE
  466 CONTINUE
C
      WRITE(6,551)
  551 FORMAT('0',10X,'THE ''V'' MATRIX IS'///)
      CALL WRITE(N,V)
      WRITE(6,553)
  553 FORMAT('0',10X,'THE ''F'' MATRIX IS'///)
      DO 595 I=1,N
      WRITE(6,555) (F(I,J,1), J=1,N)
  595 CONTINUE
      WRITE(6,596)
  596 FORMAT(///35X,'********************',20X,'********************')
  555 FORMAT(//10X,5(D20.13,2X))
C
      RETURN
      END
```

```
      SUBROUTINE QTILDA(N)
      IMPLICIT REAL*8 (A-H,O-Z), INTEGER (I-N,$)
C
      DIMENSION E1(21), E2(21), CD(20,20), F(20,20,20), QP(20,20),
     1P1(20,20), HH1(20,20), HH2(20,20)
C
      COMMON HH1 /ONE/ E1, E2 /TWO/ QP, P1 /THREE/ CD,F /FOUR/ HH2
C
C
C
      QP(1,1) = (E2(1) - E1(1))/(CD(1,N)*F(1,N,1))
      IF(N.EQ.1) RETURN
      CALL QPIJ(N,1)
C
      IF(N.GE.3) GO TO 1600
C
      T = 0.0
      DO 310 I=1,2
      T = T + HH2(I,I)**2
      T = T - HH1(I,I)**2
  310 CONTINUE
      T1 = HH2(1,2)*HH2(1,1)
      T2 = HH1(1,2)*HH1(1,1)
      QP(2,2) = QP(1,1) + T + 2.0*(T1-T2)
      GO TO 3011
C
 1600 NP1D2 = (N+1)/2
      DO 311 IEYE=2,NP1D2
      M = 2*(IEYE-1)
      COUNT = 0.0D+00
      NMM = N-M
      DO 312 J=NMM,N
      JCONJ = N-J+NMM
      NMJP1 = N-J+1
      DO 313 I=1,NMJP1
      ED = 0.0D+00
      NMJCP1.= N-JCONJ+1
      DO 314 K=1,NMJCP1
      IF((K.EQ.IEYE).AND.(I.EQ.IEYE)) GO TO 314
      ED = ED + (F(K,JCONJ,I)*QP(I,K))
  314 CONTINUE
      FD = CD(I,J)*ED
      COUNT = COUNT + FD
  313 CONTINUE
  312 CONTINUE
C
      NMIP1 = N-IEYE+1
      QP(IEYE,IEYE) = (-COUNT+E2(IEYE)-E1(IEYE))/(CD(IEYE,NMIP1)*
```

```
     1F(IEYE,NMIP1,IEYE))
C
      CALL QPIJ(N,IEYE)
C
  311 CONTINUE
C
C
C
      NDIV2 = N/2
      N1D2P1 = NP1D2 + 1
      IF(NDIV2.EQ.NP1D2) GO TO 1700
C
C
C
      DO 411 IEYE=N1D2P1,N
      LK = 2*(IEYE-NP1D2)
      COST = 0.0D+00
      NMLK = N-LK
      DO 412 J=1,NMLK
      JCON = NMLK-J+1
      NMJP1A = N-J+1
      DO 413 I=1,NMJP1A
      HD = 0.0D+00
      NMJCON = N-JCON+1
      DO 414 K=1,NMJCON
      IF((K.EQ.IEYE).AND.(I.EQ.IEYE)) GO TO 414
      HD = HD + (F(K,JCON,I)*QP(I,K))
  414 CONTINUE
      GD = CD(I,J)*HD
      COST = COST + GD
  413 CONTINUE
  412 CONTINUE
C
      NIP1 = N-IEYE+1
      QP(IEYE,IEYE) = (-COST+E2(IEYE)-E1(IEYE))/(CD(IEYE,NIP1)*
     1F(IEYE,NIP1,IEYE))
C
      IF(IEYE.EQ.N) GO TO 3011
      CALL QPIJ(N,IEYE)
C
  411 CONTINUE
C
C
C
 1700 DO 511 IEYE=N1D2P1,N
      KK = 2*(IEYE-NP1D2) - 1
      NMKK = N-KK
      COVER = 0.0D+00
```

```
      DO 512 J=1,NMKK
      JCONG = NMKK-J+1
      NMJP1B = N-J+1
      DO 513 I=1,NMJP1B
      RD = 0.0D+00
      NJP1 = N-JCONG+1
      DO 514 K=1,NJP1
      IF((K.EQ.IEYE).AND.(I.EQ.IEYE)) GO TO 514
      RD = RD + (F(K,JCONG,I)*QP(I,K))
  514 CONTINUE
      SD = CD(I,J)*RD
      COVER = COVER + SD
  513 CONTINUE
  512 CONTINUE
C
      NIEP1 = N-IEYE+1
      QP(IEYE,IEYE) = (-COVER+E2(IEYE)-E1(IEYE))/(CD(IEYE,NIEP1)*
     1F(IEYE,NIEP1,IEYE))
C
      IF(IEYE.EQ.N) GO TO 3011
      CALL QPIJ(N,IEYE)
C
  511 CONTINUE
C
 3011 WRITE(6,3010)
 3010 FORMAT('1',10X,'THE ''Q-TILDA'' MATRIX IS'///)
      CALL WRITE(N,QP)
C
      RETURN
      END
```

```
      SUBROUTINE QPIJ(N,IVAL)
      IMPLICIT REAL*8 (A-H,O-Z), INTEGER (I-N,$)
C
      DIMENSION QP(20,20), P1(20,20)
C
      COMMON /TWO/ QP, P1
C
C
C
      NL1 = N-1
      IF(NL1.EQ.IVAL) GO TO 774
      IA1 = IVAL + 1
      DO 771 J=IA1,NL1
      SUM = 0.0D+00
      JL1 = J-1
      DO 772 K=1,JL1
      SUM = SUM + ((P1(K,J)/P1(J,J))*QP(IVAL,K))
  772 CONTINUE
      QP(IVAL,J) = -SUM
      QP(J,IVAL) = -SUM
  771 CONTINUE
C
  774 PART = 0.0D+00
      DO 773 L=1,NL1
      PART = PART + QP(IVAL,L)
  773 CONTINUE
      QP(IVAL,N) = -PART
      QP(N,IVAL) = -PART
C
      RETURN
      END
```

```
      SUBROUTINE WRITE(N,RHO)
      IMPLICIT REAL*8 (A-H,O-Z), INTEGER (I-N,$)
C
      DIMENSION RHO(20,20)
C
C
C
      DO 7200 K=1,N,5
      M = K+4
      IF(M.GT.N) M=N
      DO 7201 I=1,N
      WRITE(6,7500) (RHO(I,J),J=K,M)
 7201 CONTINUE
      IF(M.EQ.N) GO TO 7202
 7200 CONTINUE
C
 7202 CONTINUE
C
 7500 FORMAT(//10X,5(D20.13,2X))
      WRITE(6,7501)
 7501 FORMAT(///35X,'********************',20X,'********************')
C
      RETURN
      END
C
C     FOR ACTUAL RUN THIS CARD IS  /*  IN COLUMNS 1 AND 2
C
//GO.SYSIN DD *
  4
12.00            48.00            80.00            48.00
0.00             1.00             9.00             34.00
19.00            138.00           435.00           450.00
0.00             1.00             9.00             34.00
```

LIST OF REFERENCES

1. Wall, H.S., "Polynomials Whose Zeros Have Negative Real Parts", American Mathermatics Monthly, v. 52, p. 308-322, 1945.

2. Wall, H.S., Analytic Theory of Continued Fractions, VanNostrand, 1948.

3. Frank, E., "On The Zeros of Polynomials with Complex Coefficients", American Mathematics Monthly, v. 52, p. 144-157, 1946.

4. Chen, C.F. and Shieh, L.S., "A Novel Approach to Linear Model Simplification", International Journal of Control, v. 8, No. 6, p. 561-570, December 1968.

5. Goldman, M.J., Generalized Model for Linear Systems Reduction, M.S. Thesis, University of Houston, Houston, 1973.

6. Davison, E.J., "A Method for Simplifying Linear Dynamic Systems", IEEE Transactions on Automatic Control, v. AC-11, No. 1, p. 93-101, January 1966.

7. Chen, C.F. and Haas, I.J., Elements of Control Systems Analysis, p. 239-269, 336-355, Prentice-Hall, 1968.

8. Chidambara, M.R., "Comment on 'On the Transformation to (Phase-Variable) Canonical Form'", IEEE Transactions on Automatic Control, v. AC-11, p. 607-608, July 1966.

9. Marshall, S.A., "An Approximate Method for Reducing the Order of a Linear System", Control (England), v. 10, p. 642-643, December 1966.

10. Sinha, N.K. and Pile, W., "A New Method for Reduction of Dynamic Systems", International Journal of Control, v. 14, No. 1, p. 111-118, 1971.

11. Greville, T.N.E., "The Pseudoinverse of a Rectangular or Singular Matrix and its Application to the Solution of Systems of Linear Equations", Siam Review, v. 1, No. 1, p. 38-43, January 1959.

12. Sinha, N.K. and Bereznai, G.T., "Optimum Approximation of High-order Systems by Low-order Models", International Journal of Control, v. 14, No. 5, p. 951-959, May 1971.

13. Hooke, R. and Jeeves, T.A., ""Direct Search" Solution of Numerical and Statistical Problems", Journal of the Association of Computing Machinery, v. 14, p. 212-229, 1961.

14. Chen, C.F. and Shieh, L.S., "Continued Fraction Inversion by Routh's Algorithm", IEEE Transactions on Circuit Theory, v. CT-16, No. 2, p. 197-202, May 1969.

15. Sinha, N.K. and DeBrvin, H., "Near-Optimal Control of High-order Systems Using Low-order Models", International Journal of Control, v. 17, No. 2, p. 257-262, February 1973.

16. Fellows, D.M., Sinha, N.K., and Wismath, J.C., "Reduction of the Order of Dynamic Models", Proceedings National Conference on Automatic Control, Canada, 1970.

17. Solheim, O.A., "Design of Optimal Control Systems with Prescribed Eigenvalues", International Journal of Control, v. 15, No. 1, p. 143-160, January 1972.

18. Chang, S.S.L., Synthesis of Optimum Control Systems, McGraw Hill, 1961.

19. Tyler, J.S. and Tuteur, F.B., "The use of a Quadratic Performance Index to Design Multivariable Control Systems", IEEE Transactions on Automatic Control, v. AC-11, No. 1, p. 84-92, January 1966.

20. Anderson, B.D.O. and Moore, J.B., "Linear System Optimization with Prescribed Degree of Stability", Proceedings of IEEE, v. 116, No. 12, December 1969.

21. Silverman, L.M., "Transformation of Time-Variable Systems to Canoncal (Phase-Variable) Form", IEEE Transactions on Automatic Control, v. AC-11, p. 300-303, April 1966.

22. Chen, C.F. and Shieh, L.S., "An Algebraic Method for Control System Design", International Journal of Control, v. 11, No. 5, p. 717-739, May 1970.

23. Kalman, R.E., "Mathematical Description of Linear Dynamical Systems", Siam Journal on Control, v. 1, p. 152-192, 1963.

24. Rane, D.S., "A Simplified Transformation to (Phase-Variable) Canonical Form", IEEE Transactions on Automatic Control, v. AC-11, p. 608, July 1966.

25. IBM Technical Newsletter GN26-0805, IBM System/360 and System/370 FORTRAN IV Language, 1974.

26. Gottfried, B.S., Programming with FORTRAN IV, Quantum Publishers, 1972.

BIBLIOGRAPHY

Anderson, J.H., "Geometrical Approach to Reduction of Dynamical Systems", Proceedings of IEEE, v. 114, No. 7, p. 1014-1018, 1967.

Aoki, M., "Control of Large-Scale Dynamic Systems by Aggregation", IEEE Transactions on Automatic Control, v. AC-13, No. 3, June 1968.

Arden, B.W. and Astill, K.N., Numerical Algorithms: Origins and Applications, Addison-Wesley, 1970.

Athans, M. and Falb, P.L., Optimal Control, McGraw-Hill, 1966.

Bereznai, G.T. and Sinha, N.K., "New Minimax Objective for Automated Systems Design", Electronics Letters, v. 6, No. 28, 1970.

Boyce, W.E. and DiPrima, R.C., Elementary Differential Equations and Boundary Value Problems, 3d ed., John Wiley and Sons, 1977.

Brogan, W.L., Modern Control Theory, Quantum Publishers, 1974.

Brule, J.D., "A Note on the Vandermonde Determinant", IEEE Transactions on Automatic Control, v. AC-9, p. 314-315, July 1964.

Bryson, A.E. and Ho, Y.C., Applied Optimal Control, Blaisdell, 1969.

Calovic, M.S., "An Application of Optimal Linear Regulators with Prescribed Steady States", International Journal of Control, v. 16, No. 5, p. 801-816, November 1972.

Chen, C.T., Analysis and Synthesis of Linear Control Systems, p. 104-108, Pond Woods Press, 1978.

Chidambara, M.R., "The Transformation to (Phase-Variable) Canonical Form", IEEE Transactions on Automatic Control, v. AC-10, p. 492-494, October 1965.

Desjardins, B., A User's Manual for Linear Control Programs on IBM/360, Master's Thesis, Naval Postgraduate School, Monterey, California, 1979.

Freeman, E.A. and Abbott, K.M., "Design of Optimal Linear Control Systems with Quadratic Performance Indices", Proceedings of IEEE, v. 114, No. 8, p. 1180-1183, August 1967.

Goldman, M.J., Near-Optimal Control of High Order Systems via the Cauer Forms (Feed Forward Design), Ph.D. Thesis, University of California, Los Angeles, 1976.

Golomb, M. and Shanks, M., Elements of Ordinary Differential Equations, 2d ed., McGraw-Hill, 1965.

Johnson, C.D. and Wonham, W.M., "A Note on the Transformation to Canonical (Phase-Variable) Form", IEEE Transactions on Automatic Control, v. AC-9, p. 312-313, July 1964.

Johnson, C.D. and Wonham, W.M., "Another Note on the Transformation to Canonical (Phase-Variable) Form", IEEE Transactions on Automatic Control, v. AC-11, p. 609-610, July 1966.

Kirk, D.E., Optimal Control Theory-An Introduction, Prentice-Hall, 1970.

Kolman, B., Introductory Linear Algebra-with Applications, Macmillan, 1976.

Lamba, S.S. and Rao, S.V., "Derivation of Aggregation Matrices for Simplified Models of Linear Dynamic Systems and Their Applications for Optimal Control", Proceedings of 1973 Joint Automatic Control Conference.

Melsa, J.L. and Jones, S.K., Computer Programs for Computational Assistance in the Study of Linear Control Theory, 2d ed., McGraw-Hill, 1973.

Mitra, D., "Analytical Results on the Use of Reduced Models in the Control of Linear Dynamical Systems", Proceedings of IEEE, v. 116, No. 8, p. 1439-1444, 1969.

Mufti, I.H., "On the Reduction of a System to Canonical (Phase-Variable) Form", IEEE Transactions on Automatic Control, v. AC-10, p. 206-207, April 1965.

Mufti, I.H., "Comments on 'On the Transformation to (Phase-Variable) Canonical Form'", IEEE Transactions on Automatic Control, v. AC-11, p. 609, July 1966.

Naval Postgraduate School User's Manual, 1978.

Ogata, K., Modern Control Engineering, Prentice-Hall, 1970.

Rao, S.V. and Lamba, S.S., "Suboptimal Control of Linear Systems via the Reduced Order Model of Chen and Shieh",

Rogers, R.O. and Sworder, D.D., "Suboptimal Control of Linear Dynamic Systems Using Models of Lower Dimension", AIAA Journal, v. 9, p. 1461-1467, August 1971.

Tou, J.T., "Determination of the Inverse Vandermonde Matrix", IEEE Transactions on Automatic Control, v. AC-9, p. 314, July 1964.

Tuel, Jr., W.G., "On the Transformation to (Phase-Variable) Canonical Form", IEEE Transactions on Automatic Control, v. AC-11, p. 607, July 1966.

Wright, D.J., "The Continued Fraction Representation of Transfer Functions and Model Simplification", International Journal of Control, v. 18, No. 3, p. 449-454, September 1973.

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